# Goal oriented information extraction in uniformly constructive calculi

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**Abstract.** In this paper we describe a method to extract information from constructive proofs of suitable systems using an *extractive calculus*. This method relies on the definition of *uniformly constructive calculus*, and allows to extend the family of systems for which a "good" information extraction procedure can be defined to include superintuitionistic systems for which a Normalization Theorem or a Cut-elimination Theorem does not hold. However, in the general setting we can only assure that the extraction calculus contains proofs of bounded logical complexity. In this paper we study systems for which the extraction calculus can be characterized in a goal-oriented manner. We will show that such a goal-oriented procedure can be defined for proofs in an Intuitionistic calculus of appropriate sequents. Finally we will prove that this result can be extended to the intuitionistic calculus enriched by the *Grzegorczyk Principle* and the *Descending Chain Principle*.

## 1 Introduction

In this paper we will analyze calculi from whose proofs the information can be extracted in a uniform way. They correspond to our definition of *uniformly constructive calculus*, which aims to characterize uniform extraction methods also for constructive systems neither satisfying a Normalization Theorem nor a Cut-elimination Theorem.

We think that our characterization can be useful as a foundational tool in the framework of program synthesis. In particular our methods can be used to study the computational meaning of mathematical and logical principles and their translation into program schemata (see, e.g., [1] for a similar approach).

Our notion of constructivity is based on the *disjunction property* (if a formula $A \vee B \in \mathrm{S}$ then either $A \in \mathrm{S}$ or $B \in \mathrm{S}$) and the *explicit definability property* (if a formula $\exists x A(x) \in \mathrm{S}$, then $A(t) \in \mathrm{S}$ for some term $t$ of the language).

Following the approach of the authors [1, 5–7, 14–16], this notion of constructivity (and some variants of its) is sufficient to give a formal basis to program synthesis on Abstract Data Types. In this framework, a great role is played by the constructive proofs of complex formulas representing functions to be computed or relations to be decided: here, such formulas are not used to perform

the single computations, but their proofs can be interpreted as programs to be executed by suitable computational models. On a different ground, in the area of Logic Programming, simpler logical formulas are used to directly carry out computations (driven by other logical formulas working as declarative programs); let us call *goals* such formulas according to [12, 17].

Now, also the treatment of goals in a Logic Programming attitude can be fruitfully carried out in a constructive setting, as discussed in [12, 17]. The present paper aims to link the two attitudes, providing tools to extract information from constructive proofs of goals using goal directed procedures.

The paper is so organized. After the preliminary notions (explained in next section), in Section 3 we will briefly outline our general extractive mechanism. Then, in Section 4, we will explain a goal directed extraction procedure for particular intuitionistic proofs of goals. Finally, in Section 5, we will give results allowing to apply goal oriented methods also to particular proofs of some super-intuitionistic systems with suitable inductive principles.

## 2 Preliminaries

A *(many sorted) signature* $\Sigma$ is a quadruple $\langle \mathrm{Sort}, \mathrm{Const}, \mathrm{Fun}, \mathrm{Rel} \rangle$, where Sort is the set of sort symbols, Const is a set of constant declarations, Fun and Rel are sets of function declarations and relation declarations respectively. The *well formed formulas* (*wff's* for short) of the language $\mathcal{L}_\Sigma$ are defined, as usual, starting from $\Sigma$ and the set of logical constants $\bot$, $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\forall$ and $\exists$.

Given a signature $\Sigma$ we will call *$\Sigma$-theory* any recursively enumerable set of closed wff's of the language $\mathcal{L}_\Sigma$. Hereafter, we will always consider signatures and theories satisfying the following properties: (T.1) Any signature $\Sigma$ contains a binary relation symbol $(=_s\colon s, s)$ for any $s \in \mathrm{Sort}$; (T.2) Any $\Sigma$-theory axiomatizes the relation symbol $(=_s\colon s, s)$, for every $s \in \mathrm{Sort}$, as an identity relation; (T.3) T is classically consistent, that is, no wff of the form $A \wedge \neg A$ is provable from T using Classical Logic.

Here we introduce the calculi $\mathcal{ND}_{\mathrm{Int}}$ and $\mathcal{ND}_{\mathrm{Cl}}$ for Intuitionistic and Classical Logic respectively (see [10, 18]), using the logical alphabet not including the connective $\neg$, that is $\neg A$ is taken as an abbreviation for $A \Rightarrow \bot$.

A *sequent* is an expression of the kind $\Gamma \vdash A$, where $A$ is a wff and $\Gamma$ is a finite set of wff's. Expressions of the kind $\vdash A$ and $\Gamma, B \vdash A$ will denote the sequents $\emptyset \vdash A$ with the empty set of premises and the sequent $\Gamma \cup \{B\} \vdash A$ respectively. An *initial sequent* (or *axiom*) is any sequent of the form $\Gamma, A \vdash A$.

We say that $A$ *is provable in* $\mathcal{ND}_{\mathrm{Int}}$ *from* $\Gamma$ if the sequent $\Gamma \vdash A$ is provable using the rules in Table 1. The natural deduction calculus $\mathcal{ND}_{\mathrm{Cl}}$ for first-order Classical Logic is obtained by replacing the rule $\bot_{\mathrm{Int}}$ of the calculus $\mathcal{ND}_{\mathrm{Int}}$ with the rule:

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{Cl}}$$

For the natural deduction calculi, the notions of *proof (tree)*, *end-sequent of a proof*, *subproof of a proof*, the notion of *proper parameter* as well as the notion

$$\frac{}{\Gamma, A \vdash A}\,\mathrm{Ass} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{Int}}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\mathrm{I}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\mathrm{E}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\mathrm{E}\wedge$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\,\mathrm{E}\vee$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,\mathrm{I}\Rightarrow \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \Rightarrow B}{\Gamma \vdash B}\,\mathrm{E}\Rightarrow$$

$$\frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A(x)}\,\mathrm{I}\forall \quad \text{where } y \text{ does not occur free in } \Gamma \text{ or } \forall x A(x). \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t/x)}\,\mathrm{E}\forall$$

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A(x)}\,\mathrm{I}\exists \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(y/x) \vdash C}{\Gamma \vdash C}\,\mathrm{E}\exists \quad \text{where } y \text{ does not occur free in } \Gamma,\ \exists x A(x) \text{ or } C.$$

$$\frac{}{\vdash x = x}\,\mathrm{id}_1 \qquad \frac{\Gamma \vdash p(t/x) \quad \Gamma \vdash t = t'}{\Gamma \vdash p(t'/x)}\,\mathrm{id}_2 \quad \text{where } p(x) \text{ is an atomic wff.}$$

**Fig. 1.** The calculus $\mathcal{ND}_{\mathrm{Int}}$

of *depth* of a proof $\pi$, denoted by $\mathrm{depth}(\pi)$, are defined in the usual way (see, e.g., [18]).

In the following we will define new calculi adding pseudo-natural deduction rules[1] to the above ones. Given a $\Sigma$-theory T and a pseudo-natural deduction calculus $\mathcal{ND}$, we will denote with $\mathcal{ND}(\mathrm{T})$ the calculus obtained by adding the rule

$$\frac{}{\vdash H}\,\mathrm{T}$$

to $\mathcal{ND}$ for any wff $H$ belonging to T.

Finally, given a set $\Gamma$ of wff's of $\mathcal{L}_\Sigma$, we say that $\Gamma$ is *constructive* if it satisfies the following properties of *disjunction* and *explicit definability*:

(DP) if $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$.
(ED) if $\exists x A(x) \in \Gamma$, then $A(t/x) \in \Gamma$ for some term $t$ of the language.

[1] We call pseudo-natural deduction rule any rule which does not meet the introduction/elimination paradigm, which is typical of *pure* natural deduction calculi.

## 3 The information extraction mechanism

In this section we will provide a short presentation of our mechanism to extract information from proofs, giving only the main definitions and results; for a complete discussion and a detailed presentation of all the results we refer the reader to [5–7]. We remark that, even if in this paper all the systems are presented by means of pseudo-natural deduction systems, the extraction mechanism is based on an abstract definition of a calculus allowing to treat also extraction from Gentzen-style, Tableau-style or Hilbert-style calculi.

First of all we define a *proof* on a language $\mathcal{L}_\Sigma$ as any finite object $\pi$ such that:

($\pi$.1) The (finite) set of wff's of $\mathcal{L}_\Sigma$ occurring in $\pi$ is uniquely determined and nonempty;
($\pi$.2) The sequent $\Gamma \vdash \Delta$ proved by $\pi$ is uniquely determined, where $\Gamma$ and $\Delta$ are finite sets of wff's of $\mathcal{L}_\Sigma$. $\Gamma$ (possibly empty) is the set of *assumptions* of $\pi$ while $\Delta$, which must be nonempty, is the set of *consequences* of $\pi$.

Proofs are characterized by the following attributes: $\mathrm{Seq}(\pi)$ indicates the sequent $\Gamma \vdash \Delta$ proved by $\pi$, $\mathrm{Wffs}(\pi)$ denotes the set of wff's of $\mathcal{L}_\Sigma$ occurring in $\pi$, and $\mathrm{dg}(\pi)$ denotes the *degree of* $\pi$ which is the maximum among the degrees of the wff's occurring in $\pi$. The compact notation $\pi : \Gamma \vdash \Delta$ will be used to indicate that $\mathrm{Seq}(\pi) = \Gamma \vdash \Delta$. Moreover, the degree of a sequent $\Gamma \vdash A$, denoted by $\mathrm{dg}(\Gamma \vdash \Delta)$, is the maximum among the degrees of the wff's occurring in $\Gamma \cup \Delta$.

A *calculus* on $\mathcal{L}_\Sigma$ is a pair $\mathbf{C} = (C, [\cdot])$, where $C$ is a recursive set of proofs on the language $\mathcal{L}_\Sigma$ and $[\cdot]$ is a recursive map from $C$ to the set of finite subsets of $C$ with the following properties:

($\mathbf{C}$.1) $\pi \in [\pi]$;
($\mathbf{C}$.2) For any $\pi' \in [\pi]$, $[\pi'] \subseteq [\pi]$;
($\mathbf{C}$.3) For any $\pi' \in [\pi]$, $\mathrm{dg}(\pi') \leq \mathrm{dg}(\pi)$.

The map $[\cdot]$ associates with every proof of the calculus the set of its *relevant* subproofs. We remark that conditions ($\mathbf{C}$.2) and ($\mathbf{C}$.3) are natural: the former requires that the set of subproofs of a proof also contains the subproofs of its elements; the latter requires that the degree of the subproofs of a proof must not exceed the degree of the proof.

We remark that our definition of calculus does not refer to any particular inference system, but any usual inference system (Hilbert-style, Gentzen-style, ...) is a calculus according to our definition. In particular $\mathcal{ND}_{\mathrm{Int}}$ is a calculus in this sense, where we consider $[\pi]$ to be the usual set of subproofs of $\pi$.

To simplify the notation we will identify a calculus $\mathbf{C}$ with the set of its proofs. Now, given a set of proofs $\Pi \subseteq \mathbf{C}$, we denote with $[\Pi]$ the *closure under subproofs* of $\Pi$ in the calculus $\mathbf{C}$. Namely, $[\Pi] = \{\pi' :$ there exists $\pi \in \Pi$ such that $\pi' \in [\pi]\}$. In general, $[\Pi]$ is not a recursive set of proofs. If $\Pi$ is finite then, of course, $[\Pi]$ is recursive, and hence $([\Pi], [\cdot]_{[\Pi]})$ is a calculus, where $[\cdot]_{[\Pi]}$ is the restriction of $[\cdot]$ to $[\Pi]$.

Given a calculus $\mathbf{C}$, let $\Pi \subseteq \mathbf{C}$. We define the following attributes of $\Pi$: $\mathrm{Seq}(\Pi)$ is the set of all the *sequents proved in* $\Pi$, i.e. $\mathrm{Seq}(\Pi) = \cup_{\pi \in \Pi} \mathrm{Seq}(\pi)$; $\mathrm{dg}(\Pi)$ is the *degree* of $\Pi$, i.e. $\mathrm{dg}(\Pi) = \max\{\mathrm{dg}(\pi) : \pi \in \Pi\}$, where $\mathrm{dg}(\Pi) = \infty$ if $\Pi$ contains proofs of any complexity; $\mathrm{Theo}(\Pi)$ is the set of *theorems proved in* $\Pi$, i.e. $\mathrm{Theo}(\Pi) = \{A : \vdash A \in \mathrm{Seq}(\Pi)\}$.

Now, given a rule R

$$\frac{\Gamma_1 \vdash A_1 \qquad \ldots \qquad \Gamma_n \vdash A_n}{\Delta \vdash B}\,\mathrm{R}$$

we call it an *extraction rule for* $\mathbf{C}$ (*e-rule* for short) if:

(R.1) R can be uniformly simulated in $\mathbf{C}$ w.r.t. a function $\phi_{\mathrm{R}} : \mathbf{N} \to \mathbf{N}$. That is, for every $\pi_1, \ldots, \pi_n$ in $\mathbf{C}$ such that $\pi_i : \Gamma_i \vdash A_i$ for $i = 1, \ldots, n$, there exists a proof $\pi : \Delta \vdash B \in \mathbf{C}$ such that $\mathrm{dg}(\pi)$ is less than or equal to $\max\{\phi_{\mathrm{R}}(\mathrm{dg}(\pi_1)), \ldots, \phi_{\mathrm{R}}(\mathrm{dg}(\pi_n))\}$.

(R.2) If $n = 0$, that is $R$ is a zero-premises rule, then there exists an integer $h$ such that $\Delta \vdash B$ has degree less than or equal to $h$.

(R.3) If $n > 0$ then $R$ is *non-increasing*; that is, $\mathrm{dg}(\Delta \vdash B) \leq \max\{\mathrm{dg}(\Gamma_1 \vdash A_1), \ldots, \mathrm{dg}(\Gamma_n \vdash A_n)\}$.

We remark that condition (R.1) says that R must be an admissible rule for $\mathbf{C}$, and must be simulated in a uniform way (w.r.t. the degrees) in the calculus $\mathbf{C}$.

A set $\mathcal{R}$ of e-rules for $\mathbf{C}$ is $h$-bounded ($h \in \mathbf{N}$) if, for every zero-premises rule $R \in \mathcal{R}$, the least integer $h'$ satisfying condition (R.2) is less than or equal to $h$. Moreover, if $\mathcal{R}$ is a finite set of e-rules $\{\mathrm{R}_1, \ldots, \mathrm{R}_m\}$ (where $\mathrm{R}_i$ is uniformly simulated in $\mathbf{C}$ w.r.t. $\phi_{\mathrm{R}_i}$), we define the monotone function $\phi_{\mathcal{R}} : \mathbf{N} \to \mathbf{N}$ as

$$\phi_{\mathcal{R}}(0) = \max\{\phi_{\mathrm{R}_1}(0), \ldots, \phi_{\mathrm{R}_m}(0)\}$$
$$\phi_{\mathcal{R}}(i+1) = \max\{\phi_{\mathcal{R}}(i), \phi_{\mathrm{R}_1}(i+1), \ldots, \phi_{\mathrm{R}_m}(i+1)\}$$

Examples of e-rules for $\mathcal{ND}_{\mathrm{Int}}$ are the *cut rule*

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\mathrm{CUT}$$

the *substitution rule* SUBST, and the *equality rules* $\mathrm{ID}_1$ *and* $\mathrm{ID}_2$ of Table 1 below.

It is straightforward to check that these are e-rules for $\mathcal{ND}_{\mathrm{Int}}$. In particular $\mathrm{ID}_1$ is a zero premises rule whose consequence is an atomic wff (so it is 1-bounded) while SUBST, CUT and $\mathrm{ID}_2$ are *non-increasing*.

Now, given a $h$-bounded set of e-rules for $\mathbf{C}$ and a set $\Pi$ of proofs of $\mathbf{C}$, we define the *extraction calculus for* $\mathbf{C}$ as the calculus $\mathbb{ID}(\mathcal{R}, [\Pi])$ having as axioms the sequents in $\mathrm{Seq}([\Pi])$ and the zero-premises rules in $\mathcal{R}$ and having, as the other rules, the non zero-premises rules of $\mathcal{R}$.

We remark that, according to our definition, for $\mathbb{ID}(\mathcal{R}, [\Pi])$ to be a calculus both $\mathcal{R}$ and $\Pi$ must be recursive. Actually, we will always apply this calculus starting from a recursive $\mathcal{R}$ and a finite set of proofs $\Pi$ of $\mathbf{C}$, thus being sure it is really a calculus.

In [5, 7] we have proved the following properties of the extraction calculi:

**Theorem 1.** *Let $\mathcal{R}$ be a h-bounded set of e-rules for* $\mathbf{C}$ *and let* $\Pi \subseteq \mathbf{C}$ *with* $\mathrm{dg}(\Pi) \leq k_\Pi$ *($k_\Pi \geq 0$). Then:*

1. *For every proof $\tau$ in $\mathbb{ID}(\mathcal{R}, [\Pi])$, $\mathrm{dg}(\tau) \leq \max\{h, k_\Pi\}$;*
2. *There exists a subset $\Pi'$ of* $\mathbf{C}$ *such that* $\mathrm{dg}(\Pi') \leq \max\{k_\Pi, \phi_\mathcal{R}(\max\{h, k_\Pi\})\}$ *and* $\mathrm{Seq}(\Pi') = \mathrm{Seq}(\mathbb{ID}(\mathcal{R}, [\Pi]))$.

Now, we give the definition of uniformly constructive calculus:

**Definition 1.** *Given a calculus* $\mathbf{C} = (C, [.])$, *we say that* $\mathbf{C}$ *is* uniformly constructive *if there exists a finite h-bounded set of e-rules $\mathcal{R}$ for* $\mathbf{C}$ *such that, for every* $\Pi \subseteq \mathbf{C}$, $\mathrm{Theo}(\mathbb{ID}(\mathcal{R}, [\Pi]))$ *is constructive.*

The main effect of the previous definition comes from Point (1) of Theorem 1. Indeed, this assures that, if $\pi :\vdash \exists x A(x) \in \Pi$, then we can constructively complete the information contained in the proof $\pi$ by means of the calculus $\mathbb{ID}(\mathcal{R}, [\pi])$, and the completing information can be searched in this calculus by means of an enumerative procedure only involving wff's of bounded logical complexity.

Using such a characterization the authors have shown in [6] and [7] that a wide family of systems S = T + L (where T is a mathematical theory and L is a superintuitionistic calculus) are uniformly constructive. Namely, in [7] the authors have shown that several systems S involving an Harrop theory T and superintuitionistic (intermediate) logics L are uniformly constructive. The most representative principles studied in that paper are: the *Grzegorczyk Principle* $\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B$ with $x \notin \mathrm{FV}(B)$, the *Kuroda Principle* $\forall x \neg\neg A(x) \Rightarrow \neg\neg \forall x A(x)$, the *Extended Scott Principle* $(\forall x(\neg\neg A(x) \Rightarrow A(x)) \Rightarrow \exists x(A(x) \vee \neg A(x))) \Rightarrow \exists x(\neg A(x) \vee \neg\neg A(x))$, the *Kreisel-Putnam Principle* $(\neg A \Rightarrow B \vee C) \Rightarrow (\neg A \Rightarrow B) \vee (\neg A \Rightarrow C)$ and the *Independence of Premises Principle* $(\neg A \Rightarrow \exists x B(x)) \Rightarrow \exists x(\neg A \Rightarrow B(x))$ with $x \notin \mathrm{FV}(A)$.

On the other hand, in [6] the authors have considered systems S involving theories T formalizing Abstract Data Types (according to the characterization of ADT's based on the notion of isoinitial model [3, 4, 16]) and mathematical principles such as the *Descending Chain Principle* $\exists x A(x) \wedge \forall y(A(y) \Rightarrow \exists z((A(z) \wedge z < y) \vee B)) \Rightarrow B$, the *Transfinite Induction Principle* $\forall x(\forall y(y < x \Rightarrow A(y)) \Rightarrow A(x)) \Rightarrow \forall z A(z)$, the *Markov Principle* $\forall x(A(x) \vee \neg A(x)) \wedge \neg\neg \exists x A(x) \Rightarrow \exists x A(x)$ and several kinds of inductive principles (the Descending Chain Principle and the Transfinite Induction Principle can be treated within the above definition of constructivity; on the other hand we remark that when further inductive principles are involved we may have to use a weaker version of the conditions (Dp) and (Ed), only concerning closed wff's and terms).

## 4 Hereditary Harrop formulae

In this section we will focus our attention to systems for which the calculus $\mathbb{ID}(\mathcal{R}, [\Pi])$ can be characterized in a goal oriented manner. As a matter of fact,

the proofs of uniform constructivity of all the systems discussed at the end of the previous section rely on extraction rules which are not suitable to get efficient proof search in the calculus $\mathbb{ID}(\mathcal{R}, [\Pi])$; in particular, all these results involve the CUT rule (intended as an e-rule to be applied to pseudo natural calculi not containing it). In this section we begin to investigate the possibility of searching proofs in $\mathbb{ID}(\mathcal{R}, [\Pi])$ in a *goal-directed manner*. In this sense, we want to show that extraction of information, for suitable systems, can be performed using goal-oriented procedures according to the characterization given in [12, 17]. We remark that, according to [12], the rules that can be used in such procedures are essentially the right rules of the intuitionistic sequent calculus. These rules, formalized in the usual way, are not non-increasing, that is they violate condition (R.3); however, restricting their applicability in $\mathbb{ID}(\mathcal{R}, [\Pi])$ only to derive wff's in $\mathrm{Wffs}([\Pi])$, it is possible to formulate them in such a way as to satisfy condition (R.3). In this paper, to avoid inessential details, we will concentrate on the goal oriented nature of these e-rules without presenting them in such a way as to meet condition (R.3).

Here we restrict our attention to the class of *hereditary Harrop wff's* ([12, 17]). Given a language $\mathcal{L}_\Sigma$, let $P$ and $G$ be wff's satisfying the following inductive definition, where $A$ denotes any atomic wff.

$$P := A \mid \bot \mid P_1 \wedge P_2 \mid \forall x P(x) \mid G \Rightarrow P$$
$$G := A \mid \bot \mid G_1 \wedge G_2 \mid G_1 \vee G_2 \mid \exists x G(x) \mid \forall x G(x) \mid P \Rightarrow G$$

The set of the $P$-wff's, also called *definite wff's*, will be denoted by $\mathbf{P}$, while the set of the $G$-wff's, also called *goal wff's*, will be denoted by $\mathbf{G}$. Hereafter, we will denote with $\mathcal{P}$ a set of closed definite wff's (a program according to the terminology of [12]).

In this section we will prove that the extraction of information from proofs of *goal* wff's in a calculus $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$, where $\mathcal{P}$ is a *program*, can be carried out using goal oriented rules, where the wff's involved in goals and programs are *hereditary Harrop wff's*.

Now, given a set of proofs $\Pi$ (over $\mathcal{L}_\Sigma$) and a wff $A$ of $\mathcal{L}_\Sigma$, we say that $A$ is evaluated in $\Pi$, and we write $\Pi \rhd A$, iff one of the following inductive conditions holds:

1. $A$ is an atomic wff and $\vdash A \in \mathrm{Seq}(\Pi)$;
2. $A \equiv B \wedge C$ and $\Pi \rhd B$ and $\Pi \rhd C$ and, if $B \wedge C \in \mathbf{P} \cup \mathbf{G}$, then $\vdash B \wedge C \in \mathrm{Seq}(\Pi)$;
3. $A \equiv B \vee C$ and either $\Pi \rhd B$ or $\Pi \rhd C$ and, if $B \vee C \in \mathbf{G}$, then $\vdash B \vee C \in \mathrm{Seq}(\Pi)$;
4. $A \equiv B \Rightarrow C$ and if $\Pi \rhd B$ then $\Pi \rhd C$ and, if $B \Rightarrow C \in \mathbf{P} \cup \mathbf{G}$, then $\vdash B \Rightarrow C \in \mathrm{Seq}(\Pi)$;
5. $A \equiv \forall x B(x)$ and $\Pi \rhd B(t)$ for every term $t$ of $\mathcal{L}_\Sigma$ and, if $\forall x B(x) \in \mathbf{P} \cup \mathbf{G}$, then $\vdash \forall x B(x) \in \mathrm{Seq}(\Pi)$;
6. $A \equiv \exists x B(x)$ and $\Pi \rhd B(t)$ for some term $t$ of $\mathcal{L}_\Sigma$ and, if $\exists x B(x) \in \mathbf{G}$, then $\vdash \exists x B(x) \in \mathrm{Seq}(\Pi)$.

We say that a set $\Gamma$ of wff's is evaluated in $\Pi$, and we write $\Pi \rhd \Gamma$, if $\Pi \rhd A$ holds for every $A \in \Gamma$.

From the above definition, we immediately get:

**Proposition 1.** *Let $\Pi$ be a set of proofs of $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$. If a wff $H \in \mathbf{P} \cup \mathbf{G}$ is evaluated in $\Pi$, then there exists a proof $\pi :\vdash H \in \Pi$.*

Let $\Pi$ be a set of proofs of $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$; we say that $\Pi$ is *regular*, if, for every sequent $\Delta \vdash B \Rightarrow C \in \mathrm{Seq}([\Pi])$, if $B \Rightarrow C \in \mathbf{P} \cup \mathbf{G}$, then there exists a proof $\vdash B \Rightarrow C \in \mathrm{Seq}([\Pi])$. We say that a proof $\pi$ is *regular* if $\{\pi\}$ is regular.

$$\frac{\vdash A \wedge B}{\vdash A}\,\mathbf{P}\mathrm{E}\wedge_1 \text{ with } A \wedge B \in \mathbf{P} \qquad \frac{\vdash A \wedge B}{\vdash B}\,\mathbf{P}\mathrm{E}\wedge_2 \text{ with } A \wedge B \in \mathbf{P}$$

$$\frac{\vdash \forall x A(x)}{\vdash A(t)}\,\mathbf{P}\mathrm{E}\forall \text{ with } \forall x A(x) \in \mathbf{P} \text{ and } t \text{ any term of the language} \qquad \frac{\vdash A \Rightarrow B \quad \vdash A}{\vdash B}\,\mathbf{P}\mathrm{E}\Rightarrow \text{ with } A \Rightarrow B \in \mathbf{P}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\,\mathbf{P}\mathbf{G}\mathrm{I}\wedge \text{ with } A \wedge B \in \mathbf{P} \cup \mathbf{G}$$

$$\frac{\vdash A}{\vdash A \vee B}\,\mathbf{G}\mathrm{I}\vee \text{ with } A \vee B \in \mathbf{G} \qquad \frac{\vdash B}{\vdash A \vee B}\,\mathbf{G}\mathrm{I}\vee \text{ with } A \vee B \in \mathbf{G}$$

$$\frac{\vdash A(y)}{\vdash \forall x A(x)}\,\mathbf{P}\mathbf{G}\mathrm{I}\forall \text{ with } \forall x A(x) \in \mathbf{P} \cup \mathbf{G} \text{ and } y \notin \mathrm{FV}(\forall x A(x)) \qquad \frac{\vdash A(t)}{\vdash \exists x A(x)}\,\mathbf{G}\mathrm{I}\exists \text{ with } \exists x A(x) \in \mathbf{G}$$

$$\frac{\Gamma \vdash A}{\theta\Gamma \vdash \theta A}\,\mathrm{SUBST} \qquad \frac{}{\vdash x = x}\,\mathrm{ID}_1 \qquad \frac{\vdash A(t) \quad \vdash t = t'}{\vdash A(t')}\,\mathrm{ID}_2 \text{ where } A \text{ is an atomic wff}$$

**Table 1.** The set of e-rules $\mathcal{R}_{\mathrm{hh}}$

Now, let us denote with $\mathbb{ID}_{\mathrm{hh}}(\Pi)$ the extraction calculus $\mathbb{ID}(\mathcal{R}_{\mathrm{hh}}, \Pi)$, where $\mathcal{R}_{\mathrm{hh}}$ is the set of e-rules of Table 1.

**Proposition 2.** *Let $\Pi$ be any set of proofs of $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$ and let $P$ be a definite wff. If there exists a proof $\tau :\vdash P$ in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, then $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd P$.*

*Proof.* The proof easily follows by induction on the structure of $P$ using in an essential way the rules $\mathbf{P}\mathrm{E}\wedge$, $\mathbf{P}\mathrm{E}\Rightarrow$ and $\mathbf{P}\mathrm{E}\forall$. As an example, let us consider the case $P \equiv G \Rightarrow A$. Now, if $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd G$, by Proposition 1 we have that $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ contains a proof $\tau_1 :\vdash G$ and hence, by applying the rule $\mathbf{P}\mathrm{E}\Rightarrow$, we get that $\vdash A$ has a proof in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ and thus, since $A$ is atomic, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A$.

**Lemma 1.** *Let $\Pi$ be a regular set of proofs of $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$. For every $\pi : \Gamma \vdash A$ belonging to the closure under substitution of $[\Pi]$, if $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \Gamma$ then $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A$.*

*Proof.* The proof goes by induction on $\mathrm{depth}(\pi)$. If $\mathrm{depth}(\pi) = 0$ then either the only rule applied in $\pi$ is an assumption introduction or $\mathrm{id}_1$, or an axiom introduction. In the former cases the assertion immediately follows, in the latter case $A$ is a definite wff and, by hypothesis, $\vdash A$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, hence, by Proposition 2, we get that $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A$. Now, let us suppose that the assertion holds for any proof $\pi' : \Gamma' \vdash A'$ belonging to the closure under substitution of $[\Pi]$ such that $\mathrm{depth}(\pi') \leq h$, and let us suppose that $\mathrm{depth}(\pi) = h + 1$. The proof goes by cases according to the last rule applied in $\pi$. We only treat the cases of the $\Rightarrow$-introduction and $\forall$-introduction.

*$\Rightarrow$-introduction.* In this case $\pi$ has the following form:

$$\pi : \Gamma \vdash A \equiv \dfrac{\pi_1 : \Gamma, B \vdash C}{\Gamma \vdash B \Rightarrow C}\,\mathrm{I}\Rightarrow$$

By induction hypothesis, if $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd B$ then $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd C$. Moreover, if $B \Rightarrow C \in \mathbf{P} \cup \mathbf{G}$, since $\Pi$ is regular, there exists a proof $\pi' : \vdash B \Rightarrow C$ in the closure under substitution of $[\Pi]$, and hence the sequent $\vdash B \Rightarrow C$ has a proof in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$.

*$\forall$-introduction.* In this case $\pi$ has the following form:

$$\pi : \Gamma \vdash A \equiv \dfrac{\pi_p : \Gamma \vdash B(p)}{\Gamma \vdash \forall x B(x)}\,\mathrm{I}\forall$$

Since $\pi : \Gamma \vdash B(p)$ belongs to the closure under substitution of $[\Pi]$ (with $p$ a proper parameter), we have that, for every term $t$ of the language, the proof $\pi_p[t/p] : \Gamma \vdash B(t)$ belongs to the closure under substitution of $[\Pi]$. Hence, by induction hypothesis, we have that, for every term $t$ of the language, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd B(t)$. Moreover, if $\forall x B(x) \in \mathbf{P} \cup \mathbf{G}$, since $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd B(y)$ for some variable $y \notin \mathrm{FV}(\forall x B(x))$, we have that $\vdash B(y)$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$. Thus, applying the rule **PG**I$\forall$, we get that also $\vdash \forall x B(x)$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$.

**Corollary 1.** *Let $\Pi$ be a regular set of proofs of $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$. For every $\tau : \Gamma \vdash H \in \mathbb{ID}_{\mathrm{hh}}([\Pi])$ and for every substitution $\theta$, if $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \theta\Gamma$ then $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \theta H$.*

*Proof.* The proof goes by induction on the number $h$ of rules different from SUBST applied in $\tau$. If $h = 0$ then $\tau : \Gamma \vdash H$ is obtained by applying a (possibly empty) sequence of SUBST to a sequent in $\mathrm{Seq}([\Pi])$. Hence, there exist a proof $\pi' : \Gamma' \vdash H'$ belonging to $[\Pi]$ and a substitution $\theta'$ such that $\theta'\Gamma' \vdash \theta' H' \equiv \Gamma \vdash H$; then, the sequent $\theta\Gamma \vdash \theta H$ has a proof in the closure under substitution of $[\Pi]$, and, since $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \theta\Gamma$, by Lemma 1, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \theta H$. The proof of the induction step goes on by cases according to the last rule different from SUBST applied in $\tau$.

**Theorem 2.** *Let $\Pi$ be a regular set of proofs of $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$. $\mathrm{Theo}(\mathbb{ID}_{\mathrm{hh}}([\Pi]))$ is constructive with respect to the goal wff's. That is:*

1. *If $G \equiv G_1 \vee G_2$ and $\vdash G_1 \vee G_2$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, then either $\vdash G_1$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ or $\vdash G_2$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ .*
2. *If $G \equiv \exists x G'(x)$ and $\vdash \exists x G'(x)$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, then there exists a term $t$ such that $\vdash G'(t)$ is provable in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$.*

*Proof.* Let $G \equiv G_1 \vee G_2$. If $\vdash G_1 \vee G_2 \in \mathrm{Seq}(\mathbb{ID}_{\mathrm{hh}}([\Pi]))$, since the empty set of wff's is evaluated in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, we have that $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd G_1 \vee G_2$. Hence, by definition, $\vdash G_1 \in \mathrm{Seq}(\mathbb{ID}_{\mathrm{hh}}([\Pi]))$ or $\vdash G_2 \in \mathrm{Seq}(\mathbb{ID}_{\mathrm{hh}}([\Pi]))$. The case of $G \equiv \exists x G'(x)$ is similar.

This means that given a regular set of proofs $\Pi \subseteq \mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$ containing a proof of a goal $G$ either of the kind $\exists x G'(x)$ or $G_1 \vee G_2$, the calculus $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ allows to constructively evaluate $G$; moreover the search for the proof can be carried out in an almost *goal directed* way. Indeed, the non goal oriented rules involved in this calculus either concern the decomposition of definite wff's deriving from the program $\mathcal{P}$ or are axioms directly extracted from the original proof, that is they belong to $\mathrm{Seq}([\Pi])$. We remark that the calculus $\mathbb{ID}(\mathcal{R}, [\Pi])$ also allows to evaluate the goals not involving disjunctions and existential quantifiers.

## 5 Extending $\mathcal{ND}_{\mathrm{Int}}$

In this section we will extend the result of the previous section to systems including the following principles:

$$\text{(Grz)}\quad \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B \quad \text{with } x \notin \mathrm{FV}(B)$$

$$\text{(DCP)}\quad \exists x A(x) \wedge \forall y(A(y) \Rightarrow \exists z((A(z) \wedge z < y) \vee B)) \Rightarrow B$$

Here (Grz) is *Grzegorczyk Principle*, whose addition to intuitionistic logic gives rise to a well-known intermediate constructive logic we call *Grzegorczyk Logic*, see, e.g., [5, 7, 9, 11]; for explanations of cut-free sequent calculi capturing Grzegorczyk Logic the reader is referred to [8, 13], while maximal intermediate constructive logics extending Grzegorczyk Logic are treated in [2]. (DCP) is the *Descending Chain Principle*, and has been studied in relation with program synthesis, see [1, 6, 7].

These principles can be formalized by the pseudo natural deduction rules of Table 2. We will denote with $\mathcal{ND}_{\mathrm{Int}^+}$ the calculus obtained by adding the rules Grz and DCP to $\mathcal{ND}_{\mathrm{Int}}$.

Given a set $\mathcal{P}$ we say that $\mathcal{ND}_{\mathrm{Int}^+}(\mathcal{P})$ *contains the rule* DCP *in the adequate context* if: (1) the language $\mathcal{L}_\Sigma$ contains a binary relation symbol $<$ axiomatized by $\mathcal{P}$ as an irreflexive and transitive relation (this can be done with definite wff's); (2) There exists a (classical) $\Sigma$-structure $\mathfrak{M}$ such that $\mathfrak{M} \models \mathcal{P} \cup \{(\mathrm{DCP})\}$ and the relation $<^{\mathfrak{M}}$ (that is, the interpretation in the structure $\mathfrak{M}$ of the relation symbol $<$) is *well founded*.

Now, we prove for $\mathcal{ND}_{\mathrm{Int}^+}(\mathcal{P})$ a result similar to the one stated in the previous section for $\mathcal{ND}_{\mathrm{Int}}(\mathcal{P})$.

$$\frac{\Gamma \vdash \forall x(A(x) \vee B)}{\Gamma \vdash \forall x A(x) \vee B}\,\mathrm{Grz}\ x \notin \mathrm{FV}(B)$$

$$\frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(y) \vdash \exists z(A(z) \wedge z < y) \vee B}{\Gamma \vdash B}\,\mathrm{DCP}$$

**Table 2.** The rules Grz and DCP

**Lemma 2.** *Let $\Pi$ be a regular set of proofs of $\mathcal{ND}_{\mathrm{Int}^+}(\mathcal{P})$. For every $\pi : \Gamma \vdash A$ belonging to the closure under substitution of $[\Pi]$, if $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \Gamma$ then $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A$.*

*Proof.* The proof goes along the lines of the proof of Lemma 1. We only deal with the cases corresponding to the rules Grz and DCP.

*Rule* Grz.

$$\pi : \Gamma \vdash H \equiv \frac{\pi_1 : \Gamma \vdash \forall x(A(x) \vee B)}{\Gamma \vdash \forall x A(x) \vee B}\,\mathrm{Grz}$$

If $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \Gamma$ then, by induction hypothesis, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \forall x(A(x) \vee B)$. Let us consider any term $t$ of the language. By definition we have that $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A(t/x) \vee B$ and this implies that either $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A(t/x)$ or $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd B$. In the latter case, if $\forall x A(x) \vee B \in \mathbf{G}$, by induction hypothesis $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ contains a proof of $\vdash B$; thus, applying the rule $\mathbf{G}\mathrm{I}\vee$, we get that it also contains a proof of $\vdash \forall x A(x) \vee B$. Otherwise, if $B$ is not evaluated in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, we deduce that, for every term $t$, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A(t/x)$. Moreover, if $\forall x A(x) \vee B \in \mathbf{G}$, by induction hypothesis, $\mathbb{ID}_{\mathrm{hh}}([\Pi])$ contains a proof of $\vdash A(y)$ where $y \notin \mathrm{FV}(\forall x A(x))$. Hence, by applying the rules $\mathbf{PG}\mathrm{I}\forall$ and $\mathbf{G}\mathrm{I}\vee$ to this proof, we get that both $\forall x A(x)$ and $\forall x A(x) \vee B$ have a proof in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$. In both cases, we deduce that $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \forall x A(x) \vee B$.

*Rule* DCP.

$$\frac{\pi_1 : \Gamma \vdash \exists x A(x) \qquad \pi_2 : \Gamma, A(y) \vdash \exists x(A(x) \wedge x < y) \vee B}{\Gamma \vdash B}\,\mathrm{DCP}$$

Let us suppose that $B$ is not evaluated in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$. Since, by induction hypothesis, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \exists x A(x)$, there exists a term $t_0$ such that $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A(t_0/x)$. By assuming the usual conventions on proper parameters, we have that $\pi_2[t_0/y]$ is a proof of the sequent $\Gamma, A(t_0/y) \vdash \exists x(A(x) \wedge x < t_0) \vee B$ and belongs to the closure under substitution of $[\Pi]$. Thus, by induction hypothesis, $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd \exists x(A(x) \wedge x < t_0) \vee B$. Since $B$ is not evaluated in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$, there exists a

term $t_1$ such that both $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd A(t_1)$ and $\mathbb{ID}_{\mathrm{hh}}([\Pi]) \rhd t_1 < t_0$. Iterating this argument, we can find an infinite sequence $t_0, t_1, \ldots, t_n, \ldots$ of (possibly not closed) terms of the language such that $t_1 < t_0$ , $t_2 < t_1$ , $\ldots$, $t_{n+1} < t_n$ , $\ldots$ are evaluated in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$. Since all the rules in $\mathcal{R}_{\mathrm{hh}}$ can be simulated in $\mathcal{ND}_{\mathrm{Int}^+}(\mathcal{P})$ and this calculus is contained in the corresponding classical calculus $\mathcal{ND}_{\mathrm{Cl}}(\mathcal{P})$, we get that any of the wff's $t_1 < t_0$, $t_2 < t_1$, $\ldots$, $t_{n+1} < t_n$, $\ldots$is a theorem of this calculus. Therefore, by the Soundness Theorem for the classical calculus, we have that these wff's are valid in every classical model $\mathfrak{M}$ of $\mathcal{P} \cup \{\mathrm{DCP}\}$; this means that the relation $<^{\mathfrak{M}}$ contains an infinite descending chain in any such a model $\mathfrak{M}$. But since, by hypothesis, $\mathcal{ND}_{\mathrm{Int}^+}(\mathcal{P})$ contains DCP in the adequate context, this yields a contradiction; hence $B$ must be evaluated in $\mathbb{ID}_{\mathrm{hh}}([\Pi])$.

Proceeding as in the previous section we get:

**Theorem 3.** *Let $\Pi$ be a regular set of proofs of $\mathcal{ND}_{\mathrm{Int}^+}(\mathcal{P})$.* $\mathrm{Theo}(\mathbb{ID}_{\mathrm{hh}}([\Pi]))$ *is constructive with respect to the goal wff's.*

As compared with results involving purely intuitionistic systems enriched by "weak" axioms (such as the definite wff's), the above result allows to work with reasonably powerful calculi, where also induction principles are allowed.

## 6 Further work

The paper is meant as a starting point for a further investigations merging our attitude on uniformly constructive systems with the one of the uniform proofs of [12, 17]. We plan to extend our results to formal systems including other superintuitionistic principles and inductive rules, among which ordinary induction (which is stronger than the Descending Chain Rule presented in this paper if $<$ is taken as the usual order on the natural numbers). Also, we have in mind to extend our treatment to non fully constructive systems (i.e., to *semiconstructive* systems in the sense of [6, 7]), in order to provide a notion of goal oriented extraction of information from significant fragments of Classical Logic.

## References


1. A. Avellone, M. Ferrari, and P. Miglioli. Synthesis of programs in abstract data types. In *8th International Workshop on Logic-based Program Synthesis and Transformation*, volume 1559, pages 81–100. Springer-Verlag, 1999.
2. A. Avellone, C. Fiorentini, P. Mantovani, and P. Miglioli. On maximal intermediate predicate constructive logics. *Studia Logica*, 57:373–408, 1996.
3. A. Bertoni, G. Mauri, and P. Miglioli. On the power of model theory to specify abstract data types and to capture their recursiveness. *Fundamenta Informaticae*, IV.2, 1983.

4. A. Bertoni, G. Mauri, P. Miglioli, and M. Wirsing. On different approaches to abstract data types and the existence of recursive models. *EATCS bulletin*, 9:47–57, 1979.
5. M. Ferrari. *Strongly Constructive Formal Systems*. PhD thesis, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, Italy, 1997. Available at `http://dotto.usr.dsi.unimi.it/~ferram`.
6. M. Ferrari, C. Fiorentini, and P. Miglioli. Extracting information from intermediate T-systems. *IMLA99: Intuitionistic Modal Logics and Application*, Available at `http://dotto.usr.dsi.unimi.it/~ferram`.
7. M. Ferrari, P. Miglioli, and M. Ornaghi. On uniformly constructive and semiconstructive formal systems. Submitted to *Annals of Pure and Applied Logic*, 1999.
8. C. Fiorentini and P. Miglioli. A cut-free sequent calculus for the logic of constant domains. Technical report, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, 1999.
9. D.M. Gabbay. *Semantical Investigations in Heyting's Intuitionistic Logic*. Reidel, Dordrecht, 1981.
10. G. Gentzen. Investigations into logical deduction. In M.E. Szabo, editor, *The Collected Works of Gerhard Gentzen*, pages 68–131. North-Holland, 1969.
11. S. Görnemann. A logic stronger than intuitionism. *Journal of Symbolic Logic*, 36:249–261, 1971.
12. J. Harland. A proof-theoretic analysis of goal-directed provability. *Journal of Logic and Computation*, 4(1):69–88, 1994.
13. R. Kashima and T. Shimura. Cut-Elimination Theorem for the Logic of Constant Domains. *Mathematical Logic Quarterly*, 40:153–172, 1994.
14. P. Miglioli, U. Moscato, and M. Ornaghi. Constructive theories with abstract data types for program synthesis. In D.G. Skordev, editor, *Mathematical Logic and its Applications*, pages 293–302. Plenum Press, New York, 1988.
15. P. Miglioli, U. Moscato, and M. Ornaghi. Semi-constructive formal systems and axiomatization of abstract data types. In J. Diaz and F. Orejas, editors, *TAPSOFT'89*, pages 337–351. Springer-Verlag, LNCS, 1989.
16. P. Miglioli, U. Moscato, and M. Ornaghi. Abstract parametric classes and abstract data types defined by classical and constructive logical methods. *Journal of Symbolic Computation*, 18:41–81, 1994.
17. D. Miller, G. Nadathur, F. Pfenning, and A. Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.
18. D. Prawitz. *Natural Deduction*. Almquist and Winksell, 1965.