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# Extracting information from intermediate T-systems

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**Abstract**

In this paper we will study the problem of uniformly extracting information from constructive and semiconstructive calculi. We will define an information extraction mechanism and will explain several examples of systems to which such a mechanism can be applied. In particular, we will give as examples some families of effective subsystems of a wide class of very large intermediate theories, we call **T**-systems. These large **T**-systems, even if ineffective and semantically defined, provide a uniform and fruitful framework where to analyze the possible combinations in a uniformly constructive context of mathematical and super-intuitionistic logical principles.

**Keywords**: intermediate constructive systems, information extraction

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# 1 Introduction

It is well known that formal proofs can be used for program synthesis and program verification, and this essentially depends on the availability of an *information extraction mechanism* allowing to capture in an *uniform way* the implicit algorithmic content of a proof. As it is well known (see, e.g., [Goto, 1979; Martin-Löf, 1982]) such a uniform mechanism can be defined for constructive calculi enjoining a Normalization Theorem or a Cut-elimination Theorem; also suitable fragments of classical calculi can be considered (see, e.g., [Murthy, 1990; Parigot, 1993]).

In this paper we will analyze two kinds of calculi from whose proofs the information can be extracted in a uniform way. The first one corresponds to our definition of *uniformly constructive calculus*, which aims to characterize uniform extraction methods also for constructive systems neither satisfying a Normalization Theorem nor a Cut-elimination Theorem. The second one corresponds to our notion of *uniformly semiconstructive calculus*, which intends to provide a general framework where to study the extraction of information from classical proofs. In this way, considering also the notions of *constructive* and *semiconstructive* calculus (which disregard uniformity properties), we can give an accurate classification of important aspects related to a wide family of systems.

Here, we will focus our investigation on the notion of *(intermediate)* $\mathbf{T}$*-system*, which can be seen as an extension to non purely logical systems (i.e., systems involving mathematical theories) of the well known notion of intermediate logic [Avellone et al., 1996; Ono, 1972] (see §3 for the formal definition). Intermediate $\mathbf{T}$-systems (with $\mathbf{T}$ ranging in the family of first order theories) have been only occasionally considered in literature; in this paper we also aim to propose them with some systematic attitude, in particular as concerns the constructive and semiconstructive $\mathbf{T}$-systems, taking into account aspects of maximality and effectiveness (the latter being intended as synonymous with "recursive axiomatizability").

Our notion of constructive $\mathbf{T}$-system is based on the *disjunction property* (if a closed wff $A \vee B$ belongs to the system, then either $A$ or $B$ belongs to the system) and the *explicit definability property* (if a closed wff $\exists x A(x)$ belongs to the system, then $A(t)$ belongs to the system for some closed term $t$ of the language). On the other hand, we consider a $\mathbf{T}$-system $\mathbf{S}$ semiconstructive if it satisfies the *weak disjunction property* (if a closed wff $A \vee B$ belongs to $\mathbf{S}$ either $A$ or $B$ belongs to the corresponding classical theory $\mathbf{T} \oplus \mathbf{Cl}$) and the *weak explicit definability property* (if a closed wff $\exists x A(x)$ belongs to $\mathbf{S}$ then $A(t)$ belongs to the corresponding classical theory $\mathbf{T} \oplus \mathbf{Cl}$ for some closed term $t$).

According to our approach to program synthesis ([Avellone et al., 1999; Miglioli et al., 1988; Miglioli et al., 1989; Miglioli et al., 1994]), the paper is also interested in providing a significant basis to extract information from constructive $\mathbf{T}$-systems specifying Abstract Data Types. In this sense, if $\mathbf{T}$ is a theory completely formalizing an Abstract Data Type (according to the characterization of Abstract Data Types based on the notion of *isoinitial model*, see [Bertoni et al., 1983; Bertoni et al., 1993; Bertoni et al., 1984; Bertoni et al., 1979; Miglioli et al., 1994]), the addition of $\mathbf{T}$ to a deductive apparatus $\mathbf{L}$ gives rise to a recursively axiomatizable and semiconstructive (or constructive) formal system $\mathbf{S}$. Moreover, if a formula of the kind $\forall \underline{x} \exists ! y A(\underline{x}, y)$ (respectively, a formula of the kind $\forall \underline{x}(B(\underline{x}) \vee \neg B(\underline{x})))$ can be proved in $\mathbf{S}$, then the whole formal system can be used to compute the function (respectively, the predicate) associated with such a formula (see, e.g., [Bertoni et al., 1984; Miglioli et al., 1988; Miglioli et al., 1989; Miglioli et al., 1994]).

But, if the formal system **S** does not satisfy further properties, the algorithm to compute the function (the predicate) can be only based on an enumeration of the theorems of **S**; hence it is highly inefficient. On the other hand, if a Normalization Theorem (or a Cut-elimination Theorem) holds for **S**, then one can define a primitive-recursive computational model allowing to directly interpret any proof of **S** as a program (this computational model is described, e.g., in [Goto, 1979; Martin-Löf, 1982], and is known in literature as *proofs-as-programs* paradigm).

A deeper discussion on the advantages and the limits of the computational model involved in Normalization is out of the scope of this paper (for a more extensive discussion, see [Miglioli and Ornaghi, 1981]). The aspect we want to point out here is that the framework where the Normalization Theorems hold is too narrow, essentially coinciding (disregarding the non-constructive classical systems) with a family of purely intuitionistic calculi. In this sense, the assumption that the calculi enjoying Normalization (or Cut-elimination) exhaust the family of the calculi whose proofs can be *reasonably* interpreted as programs is quite reductive. Indeed, there is a great number of constructive and semiconstructive calculi which are, according to us, quite reasonable candidates to be included in this family, even if they cannot be reasonably seen as normalizable. Calculi of this kind contain interesting mathematical and intermediate logical principles which hardly can be handled in an attitude oriented to Normalization, even if they have, without any doubt, a clear algorithmic content (see, e.g., [Avellone et al., 1999]).

Thus, one of our goals is to provide tools to significantly extend the field of applications of the traditional proof-theoretic techniques, yet providing a good paradigm of proofs-as-programs both in the context of constructive formal systems and in the context of semiconstructive formal systems.

The paper is organized as follows. In §2 we will introduce the basic definitions and calculi we will use in the paper. In §3 we will discuss the fundamental notions of **T**-system and theory completely formalizing an Abstract Data Type, also providing some results of maximality and ineffectiveness. In §4 we will give the main results on our information extraction mechanism, and the fundamental definitions of uniformly constructive and semiconstructive calculus. Finally, in §5 and 6 we will present several examples of uniformly constructive and semiconstructive calculi.

# 2 Preliminaries

A *many sorted signature* is any quadruple $\mathscr{A} = \langle \mathrm{Sort}, \mathrm{Const}, \mathrm{Fun}, \mathrm{Rel} \rangle$, where: Sort is a set of *sort symbols*, we denote by $s, s_1, s_n, \ldots$; Const is a set of *constant declarations*, of the kind $c : s$, where $c$ is a constant symbol and $s$ is a sort symbol (the *sort of* $c$); Fun is a set of *function declarations*, of the kind $f : \underline{s} \to s'$, where $f$ is a *function symbol*, $\underline{s}$ is a string of sort symbols (the *arity of* $f$) and $s'$ is a sort symbol (the *sort of* $f$); Rel is a set of *relation declarations* of the kind $r : \underline{s}$, where $r$ is a *relation symbol* and $\underline{s}$ is a string of sort symbols (the *arity of* $r$).

The set of *terms* and the set of *well formed formulas* (*wff's* for short) of $\mathscr{L}_{\mathscr{A}}$ are built up in the usual way, starting from $\mathscr{A}$, a denumerable set $\mathscr{V}$ of sorted variables and the logical constants $\wedge$, $\vee$, $\to$, $\neg$, $\forall$, $\exists$. The *degree* $\mathrm{dg}(A)$ of a wff $A$ is defined in the usual way.

The notions of *free* and *bounded* individual variable, of *closed* and *open* term and wff, and the notion of *substitution* are defined as usual. Notations such as $A(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ (with $n \geq 1$) will indicate that $x_1, \ldots, x_n$ may occur free in the wff $A$ and in the term $t$ respectively, while $\mathrm{FV}(A)$ will indicate the set of all the free individual variables occurring in the wff $A$. Given any substitution $\theta$ and any wff $A$ (any term $t$), we denote with $\theta A$ (with $\theta t$) the expression obtained by (correctly) applying $\theta$ to the wff $A$ (to the term $t$). If $\Gamma$ is a set of wff's, $\theta\Gamma$ will denote the set containing the wff $\theta A$ for any $A \in \Gamma$. Finally, if $\theta$ associates with every variable a closed term of the language, we say that $\theta A$ is a *closed instance* of $A$ and that $\theta$ is a *closed substitution.*

An $\mathscr{A}$*-structure* is, as usual, a structure $\mathfrak{M} = \langle \mathbf{M}, \iota \rangle$, where $\mathbf{M} = \{\mathbf{M_s} | \mathbf{s} \in \mathrm{Sort}\}$ is a Sort-indexed family of *non-empty* sets, called the *carriers* of the sorts of $\mathscr{A}$, and $\iota$ is the interpretation function. In $\mathfrak{M}$ terms and wff's are interpreted in the usual *classical* way (see, e.g., [Chang and Keisler, 1973]).

Given a signature $\mathscr{A}$ we will call $\mathscr{A}$*-theory* any recursively enumerable set of closed wff's of the language $\mathscr{L}_{\mathscr{A}}$. Hereafter, we will always consider signatures and theories satisfying the following properties:

(i) Any signature $\mathscr{A}$ contains at least a constant declaration $c : s$ for every sort symbol $s$ of $\mathscr{A}$;

(ii) Any signature $\mathscr{A}$ contains a binary relation symbol ($=_s$: $s, s$) for every sort symbol $s$ of $\mathscr{A}$;

(iii) Any $\mathscr{A}$-theory axiomatizes the relation symbol ($=_s$: $s, s$), for every sort symbol $s$ of $\mathscr{A}$, as an identity relation;

(iv) $\mathbf{T}$ is classically consistent, that is, no wff of the form $A \wedge \neg A$ is provable from $\mathbf{T}$ using Classical Logic.

Finally, we present here a variant of the natural deduction calculi for Intuitionistic and Classical Logic due to Gentzen [Gentzen, 1969] and Prawitz [Prawitz, 1965]. Here, the logical alphabet does not include the connective $\neg$, that is $\neg A$ is taken as an abbreviation for $A \rightarrow \bot$.

We call *sequent* an expression of the kind $\Gamma \vdash A$, where $A$ is a wff and $\Gamma$ is a finite set of wff's. For the sake of simplicity, we use the following conventions: $\Gamma, \Delta \vdash A$ abbreviates $\Gamma \cup \Delta \vdash A$, $A \vdash B$ abbreviates $\{A\} \vdash B$ and $\vdash A$ abbreviates $\emptyset \vdash A$. Moreover, we call *initial sequent* or *axiom* any sequent of the form $A \vdash A$.

We will write $\Gamma \vdash_{\mathbf{Int}} A$ to indicate that a sequent $\Gamma' \vdash A$ with $\Gamma' \subseteq \Gamma$ is provable using the rules in Table 1. The natural deduction calculus $\mathscr{N}\!\mathscr{D}_{\mathbf{Cl}}$ for first-order Classical Logic is obtained by replacing the rule $\bot_{\mathrm{Int}}$ of the calculus $\mathscr{N}\!\mathscr{D}_{\mathbf{Int}}$ with the rule:

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\ \bot_{\mathrm{Cl}}$$

For the natural deduction calculi, the notions of *proof (tree)*, *end-sequent of a proof*, *subproof of a proof*, as well as the notion of *depth* of a proof $\pi$, denoted by $\mathrm{depth}(\pi)$, are defined in the usual way (see, e.g., [Prawitz, 1965; Takeuti, 1975]).

The variable $y$ in the rules I$\forall$ and E$\exists$ of Table 1 is called *proper parameter of the rule.* We call *free variable of a proof* every variable which occurs free in some wff of the proof and does not occur as a proper parameter in such a wff. It is well known that proper parameters can always be chosen in such a way that:

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\,\mathrm{W} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{Int}} \text{ where } A \text{ is an atomic wff.}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\mathrm{I}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\mathrm{E}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\mathrm{E}\wedge$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash A \vee B \quad \Delta, A \vdash C \quad \Theta, B \vdash C}{\Gamma, \Delta, \Theta \vdash C}\,\mathrm{E}\vee$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\,\mathrm{I}\rightarrow \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \rightarrow B}{\Gamma, \Delta \vdash B}\,\mathrm{E}\rightarrow$$

$$\frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A(x)}\,\mathrm{I}\forall \text{ where } y \text{ does not occur free in } \Gamma \text{ or } \forall x A(x). \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t/x)}\,\mathrm{E}\forall$$

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A(x)}\,\mathrm{I}\exists \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Delta, A(y/x) \vdash C}{\Gamma, \Delta \vdash C}\,\mathrm{E}\exists \text{ where } y \text{ does not occur free in } \Delta,\ \exists x A(x) \text{ or } C.$$

$$\frac{}{\vdash x = x}\,\mathrm{id}_1 \qquad \frac{\Gamma \vdash A(t/x) \quad \Delta \vdash t = t'}{\Gamma, \Delta \vdash A(t'/x)}\,\mathrm{id}_2 \text{ where } A(x) \text{ is an atomic wff.}$$

Table 1: The calculus $\mathscr{ND}_{\mathbf{Int}}$

(P1) Every proper parameter in a proof $\pi$ is a proper parameter of exactly one rule;

(P2) The set of proper parameters is disjoint from the set of free variables of a proof.

For any proof satisfying Conditions (P1) and (P2), the tree-structure obtained by replacing some of the free variables of the proof with terms is a well defined proof. We will write $\pi[t/x]$ to denote the proof obtained by substituting all the occurrences of the free variable $x$ in the proof $\pi$ with the term $t$, and $\theta\pi$ to denote the proof obtained by applying the substitution $\theta$ to $\pi$.

In the following we will define new calculi adding pseudo-natural deduction rules[1] to the above ones. Whenever we will introduce a rule with parameters, we will assume that the above Conditions (P1) and (P2) hold for any proof of the resulting calculus.

Finally, given a theory $\mathbf{T}$ and a pseudo-natural deduction calculus $\mathscr{ND}$, we will denote with $\mathscr{ND}(\mathbf{T})$ the calculus obtained by adding the rule

$$\frac{}{\vdash H}\,\mathbf{T}$$

to $\mathscr{ND}$ for any wff $H$ belonging to $\mathbf{T}$.

[1] We call pseudo-natural deduction rule any rule which does not meet the introduction/elimination paradigm, which is typical of *pure* natural deduction calculi.

## 3 Intermediate T-systems

Considering the (one sorted) language $\mathscr{L}$ of pure predicate calculus [Avellone et al., 1996; Ono, 1972], **Int** and **Cl** will denote the set of intuitionistically valid wff's of $\mathscr{L}$ and classically valid wff's of $\mathscr{L}$ respectively.

A *(first-order) intermediate logic* is any set of wff's **L** such that:

(i) $\mathbf{Int} \subseteq \mathbf{L} \subseteq \mathbf{Cl}$;

(ii) **L** is closed under modus ponens and generalization;

(iii) **L** is closed under predicate substitution (see, e.g., [Avellone et al., 1996; Ono, 1972] for a formal definition).

An *intermediate pseudo-logic* is any set **L** of wff's of $\mathscr{L}$ satisfying Conditions (i) and (ii), but possibly not satisfying Condition (iii) above.

If $\Gamma$ is a set of classically valid wff's of $\mathscr{L}$ (axioms) and **L** is an intermediate logic, the notation $\mathbf{L} + \Gamma$ will indicate the smallest set of wff's (which is an intermediate logic) closed under modus ponens, generalization and predicate substitutions containing **L** and the wff's (axioms) of $\Gamma$. On the other hand, the notation $\mathbf{L} \oplus \Gamma$ will denote the smallest set of wff's (which is an intermediate pseudo-logic) closed under modus ponens and generalization containing **L** and the wff's (axioms) of $\Gamma$. If $\Gamma = \{A\}$ consists of a single axiom, the notations $\mathbf{L} + A$ and $\mathbf{L} \oplus A$ will replace $\mathbf{L} + \{A\}$ and $\mathbf{L} \oplus \{A\}$ respectively.

Passing from the language $\mathscr{L}$ to the language $\mathscr{L}_{\mathscr{A}}$ generated by a many-sorted signature $\mathscr{A}$ (in the sense explained in §2, where the relation declarations of $\mathscr{A}$ are seen as *constant relation declarations*, and hence predicate substitution are not allowed in $\mathscr{L}_{\mathscr{A}}$), one can easily define $\mathbf{Int}_{\mathscr{A}}$ and $\mathbf{Cl}_{\mathscr{A}}$: they are the subsets of $\mathscr{L}_{\mathscr{A}}$ obtained by (correctly, i.e., without clashes) substituting the predicate variables (i.e., the atomic subformulas) with wff's of $\mathscr{L}_{\mathscr{A}}$ in the wff's of **Int** and **Cl** respectively. In a similar way, if **L** is a (first-order) intermediate logic (in the language $\mathscr{L}$), one defines $\mathbf{L}_{\mathscr{A}}$. On the other hand, a pseudo-logic $\mathbf{L}_{\mathscr{A}}$ will be any set of wff's of $\mathscr{L}_{\mathscr{A}}$ such that $\mathbf{Int}_{\mathscr{A}} \subseteq \mathbf{L}_{\mathscr{A}} \subseteq \mathbf{Cl}_{\mathscr{A}}$, and $\mathbf{L}_{\mathscr{A}}$ is closed under modus ponens and generalization. When $\mathscr{A}$ (and hence $\mathscr{L}_{\mathscr{A}}$) is understood, we will still indicate $\mathbf{Int}_{\mathscr{A}}$ and $\mathbf{Cl}_{\mathscr{A}}$ with **Int** and **Cl**; likewise for intermediate logics and pseudo-logics.

In this framework, let **T** be any $\mathscr{A}$-theory; we call *(intermediate)* **T**-*system* any set **S** of wff's of $\mathscr{L}_{\mathscr{A}}$ such that $\mathbf{Int} \oplus \mathbf{T} \subseteq \mathbf{S} \subseteq \mathbf{Cl} \oplus \mathbf{T}$ and **S** is closed under modus ponens and generalization.

Now, let us introduce the notions of constructive and semiconstructive set of wff's. Let $\Gamma$ be a set of wff's of $\mathscr{L}_{\mathscr{A}}$; we say that $\Gamma$ is *constructive* if it has the properties:

(Dp) : if $A \vee B \in \Gamma$ and $A \vee B$ is a closed wff, then either $A \in \Gamma$ or $B \in \Gamma$;

(Ed) : if $\exists x A(x) \in \Gamma$ and $\exists x A(x)$ is a closed wff, then $A(t/x) \in \Gamma$ for some closed term $t$ of the language.

(Dp) is called *disjunction property* while (Ed) is called *explicit definability property.*

Given $\Gamma, \Delta \subseteq \mathscr{L}_{\mathscr{A}}$ such that $\Gamma \subseteq \Delta$, we say that $\Gamma$ is *semiconstructive in* $\Delta$ iff the following properties (wDp) and (wEd) hold:

(wDp): if $A \vee B \in \Gamma$ and $A \vee B$ is a closed wff, then either $A \in \Delta$ or $B \in \Delta$.

(wEd): if $\exists x A(x) \in \Gamma$ and $\exists x A(x)$ is a closed wff, then $A(t/x) \in \Delta$ for some closed term $t$ of the language.

Given a **T**-system **S**, we simply say that it is semiconstructive if **S** is semiconstructive in $\mathbf{Cl} \oplus \mathbf{T}$.

In the following we will be interested in studying theories whose axioms belong to one of the classes of wff's defined below. Let $H$ be any quantifier free wff.

- A $\forall$-*wff* is any wff of the kind $\forall \underline{x} H$, where $\forall \underline{x}$ indicates a possibly empty list of $\forall$-quantifiers.
- An $\exists$-*wff* is any wff of the kind $\exists \underline{x} H$, where $\exists \underline{x}$ indicates a possibly empty list of $\exists$-quantifiers.
- A $\forall\exists$-*wff* is any wff of the kind $\forall \underline{x} \exists \underline{y} H$, where $\forall \underline{x}$ and $\exists \underline{y}$ indicate possibly empty lists of the corresponding quantifiers.
- A $\forall\exists\neg$-*wff* is inductively so defined: every $\forall\exists$-wff is a $\forall\exists\neg$-wff; every negated wff is a $\forall\exists\neg$-wff; if $A$ and $B$ are $\forall\exists\neg$-wff's and $C$ is any wff, $A \wedge B$, $C \rightarrow A$ and $\forall \underline{x} A$ are $\forall\exists\neg$-wff's.
- A $\forall\neg$-*wff* is defined like a $\forall\exists\neg$-wff, but starting from $\forall$-wff's and negated wff's in place of $\forall\exists$-wff's and negated wff's.
- An *Harrop-wff* (see [Troelstra, 1973]) is inductively so defined: every atomic or negated wff is an Harrop wff; if $A$ and $B$ are Harrop wff's and $C$ is any wff, then $A \wedge B$, $C \rightarrow A$ and $\forall \underline{x} A$ are Harrop wff's.

We will call $\mathscr{H}$-theory any theory **T** such that every wff $A \in \mathbf{T}$ is an $\mathscr{H}$-wff, where $\mathscr{H}$ is one of the notions defined above.

## 3.1 ADT as reachable isoinitial models

In this section we give the main definitions and results needed to introduce the formalization of *Abstract Data Types* (*ADT's* for short) as reachable isoinitial models.

Given two $\mathscr{A}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ we say that an homomorphism $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is an *isomorphic embedding* iff $\mathfrak{A}$ is isomorphic to the homomorphic image of $\psi$ in $\mathfrak{B}$ (see [Chang and Keisler, 1973] for a detailed definition of these notions). Let **T** be an $\mathscr{A}$-theory; a model $\mathfrak{I}$ of **T** is *isoinitial* iff, for every model $\mathfrak{M}$ of **T**, there exists a unique isomorphic embedding $\psi : \mathfrak{I} \rightarrow \mathfrak{M}$.

The formalization of ADT's we will use here is based on the notion of isoinitial model. In literature there is another well known model theoretic notion which has been used to formalize ADT's, namely that of initial model, where a model $\mathfrak{K}$ of a theory **T** is *initial* iff, for every model $\mathfrak{M}$ of $\mathfrak{T}$, there exists a unique homomorphism $\psi : \mathfrak{K} \rightarrow \mathfrak{M}$. For a detailed discussion on the approach based on initial models we refer the reader to [Wirsing, 1990], while, for a detailed discussion on the approach based on isoinitial models and for an extensive comparison between the two approaches, we refer the reader to [Bertoni et al., 1983; Bertoni et al., 1993; Bertoni et al., 1984; Bertoni et al., 1979; Miglioli et al., 1994]. It is worth mentioning that, in the full first order frame, there are theories without initial and isoinitial models. Comparing the two notions, we have that both capture "abstractness", in the paradigmatic sense of the literature on ADT's; that is, an isoinitial model (respectively, an initial model) of a first order theory $\mathbf{T}$ (if it exists) is unique up to isomorphisms.

According to the quoted literature, the characterization of an ADT as an isoinitial model seems to be well justified on the theoretical ground. In this line, we give the following definitions:

**Definition 3.1** *Given an $\mathscr{A}$-theory $\mathbf{T}$ we say that:*

*(i)* $\mathbf{T}$ *formalizes and ADT $\mathfrak{I}$ iff $\mathfrak{I}$ is an isoinitial model of* $\mathbf{T}$.

*(ii)* $\mathbf{T}$ *completely formalizes an ADT $\mathfrak{I}$ iff $\mathfrak{I}$ is a reachable isoinitial model of* $\mathbf{T}$.

We recall that an $\mathscr{A}$-structure $\mathfrak{M}$ is *reachable* iff, for every sort $s$, all the elements of its carrier are denoted by closed terms of the language $\mathscr{L}_{\mathscr{A}}$. We also recall (as explained, e.g., in [Miglioli et al., 1994]) that any theory $\mathbf{T}$ formalizing an ADT $\mathfrak{I}$ can be extended (with the addition of a recursive set of new constants and a recursive set of definitional axioms) into a theory $\mathbf{T}'$ completely formalizing an ADT $\mathfrak{I}'$ which is an expansion (up to the new language) of of $\mathfrak{I}$.

Now, we say that an $\mathscr{A}$-theory $\mathbf{T}$ is *atomically complete* iff $A \in \mathbf{Cl}{\oplus}\mathbf{T}$ or $\neg A \in \mathbf{Cl}{\oplus}\mathbf{T}$ for every closed atomic wff $A$ of $\mathscr{L}_{\mathscr{A}}$. The following theorem, whose proof is implicitly given in [Bertoni et al., 1984; Miglioli et al., 1994], provides a useful criterion to study isoinitiality.

**Theorem 3.2** *An $\mathscr{A}$-theory $\mathbf{T}$ completely formalizes an ADT iff $\mathbf{T}$ has a reachable model and is atomically complete.* □

Since the usual theory **PA** of Arithmetic is atomically complete and the standard model $\mathfrak{N}$ of **PA** is reachable (in the language of **PA**), the previous theorem allows to assert that **PA** completely formalizes an ADT.

Theorem 3.2 also allows to prove that, for every set $\mathscr{C}$ of constant symbols and function symbols, the term-algebra generated by $\mathscr{C}$ is an isoinitial model for the theory $\mathbf{T}(\mathscr{C})$ containing the identity theory of $\mathscr{C}$ and the injectivity axioms of $\mathscr{C}$; in this framework, also induction principles can be added (see [Miglioli et al., 1994]). The injectivity axioms state that different closed terms represent different elements in every model. Identity and injectivity are sufficient to obtain the isoinitiality result, but it is useful to introduce also various induction principles in order to get a powerful $\mathbf{T}$-system.

Now, let us denote with **IKa** the intermediate pseudo-logic $\mathbf{Int} \oplus (\mathrm{Kur}) \oplus (\mathrm{At})$, where (At) and (Kur) are the following principles (namely, sets of wff's having the form indicated below):

$$
\begin{array}{ll}
(\mathrm{At}) & \neg\neg A \to A \quad \text{with } A \text{ an atomic wff} \\
(\mathrm{Kur}) & \forall x \neg\neg A(x) \to \neg\neg \forall x A(x)
\end{array}
$$

Moreover, given a signature $\mathscr{A}$ and a relation declaration $r : \underline{s}$ in $\mathscr{A}$, the *canonical constraint associated with* $r : \underline{s}$, denoted by $\mathrm{CC}(r : \underline{s})$, is the wff $\forall \underline{x}(r(\underline{x}) \vee \neg r(\underline{x}))$. Given an $\mathscr{A}$-theory $\mathbf{T}$, by the *canonical extension of* $\mathbf{T}$, denoted by $\mathrm{CC}(\mathbf{T})$, we mean the theory $\mathbf{T} \cup \{\mathrm{CC}(r : \underline{s}) |\ r : \underline{s} \in \mathscr{A}\}$.

Using Theorem 3.2, we can prove the following sufficient condition for an $\mathscr{A}$-theory $\mathbf{T}$ to completely formalize an ADT (see [Miglioli et al., 1994]):

**Theorem 3.3** *Let* **T** *and* **L** *be respectively an* $\mathscr{A}$*-theory and a pseudo-logic such that:*

1. **T** *has a reachable model;*
2. $\mathbf{L} \oplus \mathbf{T}$ *is semiconstructive;*
3. *For every relation declaration* $r : \underline{s}$ *in* $\mathscr{A}$*,* $\textsc{cc}(r : \underline{s}) \in \mathbf{L} \oplus \mathbf{T}$*.*

*Then* **T** *completely formalizes an ADT.* □

The above sufficient condition can be made a necessary and sufficient one for ∀∃¬-theories, i.e. theories **T** containing only ∀∃¬-wff's; to do so we need the following theorem (see [Miglioli et al., 1994]):

**Theorem 3.4** *Let* **T** *be any atomically complete* ∀∃¬*-theory with a reachable model. Then* $\mathbf{IKa} \oplus \mathbf{T}$ *is constructive.* □

At this point, to get our necessary and sufficient condition, we have to take into account the canonical constraints of **T**. Of course, these wff's are quite irrelevant from the classical point of view, i.e., **T** and $\textsc{cc}(\mathbf{T})$ are classically equivalent and, since the new axioms of $\textsc{cc}(\mathbf{T})$ do not affect the class of models of **T**, **T** completely formalizes an ADT iff $\textsc{cc}(\mathbf{T})$ does. On the other hand, they are quite relevant from a constructive point of view. Indeed, any canonical constraint $\textsc{cc}(r : \underline{s})$ is a ∀-wff, hence a ∀∃¬-wff; thus, we can combine Theorem's 3.3 and 3.4 and state, in "purely constructive terms", the desired necessary and sufficient condition:

**Theorem 3.5** *Let* **T** *be any* ∀∃¬*-theory. Then* **T** *completely formalizes an ADT iff the following conditions are satisfied:*

1. **T** *has a reachable model;*
2. $\mathbf{IKa} \oplus \textsc{cc}(\mathbf{T})$ *is constructive.* □

## 3.2 The T-systems $\mathbf{Constr}_1(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T})$

Here, we explain the two constructive frameworks within which we will develop our further treatment, i.e., the **T**-systems $\mathbf{Constr}_1(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T})$. They are two very large families of constructive **T**-systems (depending on **T**) which, as we will see, are quite non effective (i.e., in general they are far from being recursively enumerable). Nevertheless, we believe, they are interesting for various reasons, among which the two following ones. First of all, they can be seen as a kind of semantical tool to single out superintuitionistic logical principles immediately giving rise (if added to **T**-systems of the form $\mathbf{Int} \oplus \mathbf{T}$) to semiconstructive **T**-systems, to be successively investigated in order to check whether they are uniformly constructive or uniformly semiconstructive. In this sense, they contain almost all the constructive superintuitionistic principles taken into account in literature (e.g., the principles (Mk), (Kur), ($KP_\vee$), ($KP_\exists$), (St) of Table 2), and some new superintuitionistic principles (e.g., the principles (wGrz), (St∃), and (DT) of Table 2). Secondly, the counterposition between $\mathbf{Constr}_1(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T})$ accounts for the most known facts of constructive incompatibility of superintuitionistic constructive principles. Indeed, principles whose simultaneous addition to intuitionistic systems is known to give rise to non constructive (even to non semiconstructive) systems (e.g., (Mk) and ($KP_\exists$)) are separated by the (maximal constructive) **T**-system $\mathbf{Constr}_1(\mathbf{T})$ and the (very large, perhaps maximal constructive) **T**-system $\mathbf{Constr}_2(\mathbf{T})$.

Other very large (families of) constructive **T**-systems (among which, provably maximal constructive ones) might be presented, which we omit for the sake of brevity. We have chosen $\mathbf{Constr}_1(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T})$ since, we believe, they have a great "heuristic content", i.e., they can be seen as a "kind of semantics" allowing to find, in a reasonably simple way, new interesting superintuitionistic principles. From this point of view, perhaps the reader will find connections between the semantics involved in $\mathbf{Constr}_2(\mathbf{T})$ (which is inspired by the original semantics of Medvedev Logic [Medvedev, 1963]) and some variants of the realizability semantics introduced in the traditional studies on the foundations of constructive mathematics (see, e.g., [Troelstra, 1973]). However, differently from that tradition, we are not interested in calculi which (even if consistent) violate the requirements involved in *classical* consistency. Thus, our constructive **T**-systems (and the related principles) in general are not *recursively realizable* (according to the most typical notions of realizability such as Kleene's 1945-realizability [Kleene, 1952]). In particular, in our attitude a principle of Table 2 such as (Kur) may assume a great importance, even if it is not realized by paradigmatic notions of realizability such as Kleene's one [Kleene, 1952].

Let **T** be an $\mathscr{A}$-theory; we say that a wff $A$ is *constructively sound* in **T** iff:

(i) $A \in \mathbf{Cl} \oplus \mathbf{T}$;

(ii) For every closed instance $\theta A$ of $A$, one of the following conditions holds:

  (a) $\theta A$ is atomic or negated;
  
  (b) $\theta A \equiv B \wedge C$, and both $B$ and $C$ are constructively sound in **T**;
  
  (c) $\theta A \equiv B \vee C$, and either $B$ is constructively sound in **T** or $C$ is constructively sound in **T**;
  
  (d) $\theta A \equiv B \rightarrow C$, and, if $B$ is constructively sound in **T**, then $C$ is constructively sound in **T**;
  
  (e) $\theta A \equiv \exists x B(x)$, and there exists a closed term $t$ such that $B(t)$ is constructively sound in **T**;
  
  (f) $\theta A \equiv \forall x B(x)$, and, for every closed term $t$, $B(t)$ is constructively sound in **T**.

Let us define the set

$$\mathbf{Constr}_1(\mathbf{T}) = \{A : A \text{ is constructively sound in } \mathbf{T}\}.$$

Then, if $\mathbf{T} \subseteq \mathbf{Constr}_1(\mathbf{T})$, $\mathbf{Constr}_1(\mathbf{T})$ is a maximal constructive and a maximal semiconstructive **T**-system, in the sense of the following theorem.

**Theorem 3.6** *Let* **T** *be an* $\mathscr{A}$*-theory such that* $\mathbf{T} \subseteq \mathbf{Constr}_1(\mathbf{T})$. *Then:*

1. $\mathbf{Constr}_1(\mathbf{T})$ *is a constructive* **T***-system.*

2. *For every semiconstructive* **T***-system* $\mathbf{S} \supseteq \mathbf{Constr}_1(\mathbf{T})$*, it holds that* $\mathbf{S} = \mathbf{Constr}_1(\mathbf{T})$.

*Proof:* (1) Clearly $\mathbf{Constr}_1(\mathbf{T}) \subseteq \mathbf{Cl} \oplus \mathbf{T}$. In order to prove that $\mathbf{Int} \oplus \mathbf{T} \subseteq \mathbf{Constr}_1(\mathbf{T})$, it suffices to show (by induction on the structure of the proofs) that, for every proof $\pi : \Gamma \vdash A$ in $\mathscr{ND}_{\mathbf{Int}}$, if $\Gamma \subseteq \mathbf{Constr}_1(\mathbf{T})$, then $A \in \mathbf{Constr}_1(\mathbf{T})$. The closure of $\mathbf{Constr}_1(\mathbf{T})$ with respect to modus ponens, generalization, (Dp) and (Ed) is immediate.

(2) If $\mathbf{T}$ is a *classically complete* theory (i.e., $H \in \mathbf{Cl} \oplus \mathbf{T}$ or $\neg H \in \mathbf{Cl} \oplus \mathbf{T}$ for every closed wff $H \in \mathscr{L}_{\mathscr{A}}$) and $\mathbf{T}$ has a reachable model, then it is not difficult to show that $\mathbf{Constr}_1(\mathbf{T}) = \mathbf{Cl} \oplus \mathbf{T}$, hence the assertion. Now, let us suppose that $\mathbf{T}$ is not classically complete $\mathbf{Constr}_1(\mathbf{T}) \subseteq \mathbf{S}$, $\mathbf{Constr}_1(\mathbf{T}) \neq \mathbf{S}$, and $\mathbf{S}$ is semiconstructive. Moreover, let $H$ be some closed wff's such that $H \in \mathbf{S}$ but $H \notin \mathbf{Constr}_1(\mathbf{T})$. Since the principle (DT) of Table 2 is in $\mathbf{Constr}_1(\mathbf{T})$, we have that for every $K \in \mathscr{L}_{\mathscr{A}}$, $H \to K \vee \neg K \in \mathbf{Constr}_1(\mathbf{T})$, which implies $H \to K \vee \neg K \in \mathbf{S}$, which implies (by modus ponens) $K \vee \neg K \in \mathbf{S}$. Taking $K$ such that $K \notin \mathbf{Cl} \oplus \mathbf{T}$ and $\neg K \notin \mathbf{Cl} \oplus \mathbf{T}$, this gives rise to a contradiction. Finally, suppose that $\mathbf{T}$ has no reachable model. Then one easily shows that there is $\exists x A(x) \in \mathscr{L}_{\mathscr{A}}$ such that $\neg\exists x A(x) \notin \mathbf{Cl} \oplus \mathbf{T}$ and, for every closed term $t$ of $\mathscr{L}_{\mathscr{A}}$, $A(t) \notin \mathbf{Cl} \oplus \mathbf{T}$ (while $\exists x A(x) \in \mathbf{Cl} \oplus \mathbf{T}$ if $\mathbf{T}$ is classically complete). Suppose that $\mathbf{Constr}_1(\mathbf{T}) \subseteq \mathbf{S}$, $\mathbf{Constr}_1(\mathbf{T}) \neq \mathbf{S}$, and $\mathbf{S}$ is semiconstructive. Then (arguing as in the previous case) $\exists x A(x) \in \mathbf{S}$, which implies that $A(t) \in \mathbf{Cl} \oplus \mathbf{T}$ for some $t$, a contradiction. □

Among the principles constructively sound (apart from (At) which is not closed under arbitrary substitution), we can mention *Kuroda Principle* (Kur), *Markov Principle* (Mk) (provided $\mathbf{T}$ has a reachable model), *Weak Grzegorczyk Principle* (wGrz), *Scott Principle* (St), *Extended Scott Principle* (St∃) and the principle (DT) of Table 2. On the other hand, in general *Kreisel-Putnam Principle* ($\mathrm{KP}_\vee$) and its predicative extension ($\mathrm{KP}_\exists$) (also known with the name (IP), [Troelstra, 1973]) are not constructively sound (but the latter fact does not hold if $\mathbf{T}$ is a complete theory with a reachable model).

| | |
|---|---|
| (At) | $\neg\neg H \to H$ with $H$ an atomic wff |
| (Kur) | $\forall x \neg\neg A(x) \to \neg\neg\forall x A(x)$ |
| ($\mathrm{KP}_\vee$) | $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$ |
| ($\mathrm{KP}_\exists$) | $(\neg A \to \exists x B(x)) \to \exists x(\neg A \to B(x))$ with $x \notin \mathrm{FV}(A)$ |
| (Mk) | $\forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \to \exists x A(x)$ |
| (wGrz) | $\forall x \neg\neg A(x) \wedge \forall x(A(x) \vee B) \to \forall x A(x) \vee B$ with $x \notin \mathrm{FV}(B)$ |
| (St) | $((\neg\neg A \to A) \to A \vee \neg A) \to (\neg A \vee \neg\neg A)$ |
| (St∃) | $(\forall x(\neg\neg A(x) \to A(x)) \to \exists x(A(x) \vee \neg A(x))) \to \exists x(\neg A(x) \vee \neg\neg A(x))$ |
| (DT) | $\exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)$ |

Table 2: Some intermediate principles

It seems to be hard to completely characterize the theories $\mathbf{T}$ such that $\mathbf{T} \subseteq \mathbf{Constr}_1(\mathbf{T})$; nevertheless, we can describe wide families of wff's which are constructively sound in any theory to which they belong.

**Theorem 3.7** *Let* $\mathbf{T}$ *be an* $\mathscr{A}$*-theory and let* $H$ *be a closed wff such that* $H \in \mathbf{Cl} \oplus \mathbf{T}$*. Then* $H \in \mathbf{Constr}_1(\mathbf{T})$ *if one of the following conditions holds:*

1. *$H$ is an Harrop-wff;*
2. *$H$ is a $\forall\neg$-wff and $\mathbf{T}$ is atomically complete;*
3. *$H$ is a $\forall\exists\neg$-wff, $\mathbf{T}$ is atomically complete and $\mathbf{T}$ has a reachable model.* □

We now introduce another kind of semantics allowing to define constructive $\mathbf{T}$-systems. Let $A$ be any closed wff; we define the set $\textsc{Ef}(A)$ of the *evaluation forms* $\hat{A}$ of $A$, by induction on the structure of $A$.

(a) $\textsc{Ef}(A) = \{A\}$ if $A$ is either atomic or negated;

(b) $\textsc{Ef}(B \wedge C) = \{\langle \hat{B}, \hat{C} \rangle : \hat{B} \in \textsc{Ef}(B) \text{ and } \hat{C} \in \textsc{Ef}(C)\}$;

(c) $\textsc{Ef}(B \vee C) = \{\langle \hat{B}, 0 \rangle : \hat{B} \in \textsc{Ef}(B)\} \cup \{\langle \hat{C}, 1 \rangle : \hat{C} \in \textsc{Ef}(C)\}$;

(d) $\textsc{Ef}(B \to C) = \{f : f \text{ is a function and } f : \textsc{Ef}(B) \to \textsc{Ef}(C)\}$;

(e) $\textsc{Ef}(\exists x B(x)) = \{\langle t, \hat{B(t)} \rangle : t \text{ is a closed term and } \hat{B(t)} \in \textsc{Ef}(B(t))\}$;

(f) $\textsc{Ef}(\forall x B(x)) = \{f : f$ is a function associating, with every closed term $t$, an element of $\textsc{Ef}(B(t))\}$.

Let $\mathfrak{M}$ be any model of $\mathbf{T}$ and let $\hat{A} \in \textsc{Ef}(A)$. We say that $\hat{A}$ holds in $\mathfrak{M}$, and we write $\mathfrak{M} \models \hat{A}$, if one of the following inductive conditions is satisfied:

(1) $\hat{A}$ is either an atomic or negated wff $A$ and $\mathfrak{M} \models A$.

(2) $\hat{A} \equiv \langle \hat{B}, \hat{C} \rangle \in \textsc{Ef}(B \wedge C)$, and $\mathfrak{M} \models \hat{B}$ and $\mathfrak{M} \models \hat{C}$.

(3) $\hat{A} \equiv \langle \hat{B}, 0 \rangle \in \textsc{Ef}(B \vee C)$, and $\mathfrak{M} \models \hat{B}$.

(4) $\hat{A} \equiv \langle \hat{C}, 1 \rangle \in \textsc{Ef}(B \vee C)$, and $\mathfrak{M} \models \hat{C}$.

(5) $\hat{A} \equiv f \in \textsc{Ef}(B \to C)$, and both the following conditions hold:

  (i). $\mathfrak{M} \models B \to C$;
  (ii). for every $\hat{B} \in \textsc{Ef}(B)$, $\mathfrak{M} \models \hat{B}$ implies $\mathfrak{M} \models f(\hat{B})$.

(6) $\hat{A} \equiv \langle t, \hat{B(t)} \rangle \in \textsc{Ef}(\exists x B(x))$, and $\mathfrak{M} \models \hat{B(t)}$.

(7) $\hat{A} \equiv f \in \textsc{Ef}(\forall x B(x))$, and both the following conditions hold:

  (i). $\mathfrak{M} \models \forall x B(x)$;
  (ii). for every closed term $t$, $\mathfrak{M} \models f(t)$.

Note that, if $\mathfrak{M} \models \hat{A}$ for some $\hat{A} \in \textsc{Ef}(A)$, then $\mathfrak{M} \models A$, while the converse needs not to be true.

Let $A$ be any closed wff and let $\hat{A} \in \textsc{Ef}(A)$; we say that $\mathbf{T} \models \hat{A}$ iff, for every model $\mathfrak{M}$ of $\mathbf{T}$, it holds that $\mathfrak{M} \models \hat{A}$. Given a wff $A$, we denote with $\forall A$ the universal closure of $A$. We define

$$\mathbf{Constr}_2(\mathbf{T}) = \{A : \mathbf{T} \models \hat{\forall A} \text{ for some } \hat{\forall A} \in \textsc{Ef}(\forall A)\}.$$

We can prove:

**Theorem 3.8** *Let* $\mathbf{T}$ *be an* $\mathscr{A}$*-theory such that* $\mathbf{T} \subseteq \mathbf{Constr}_2(\mathbf{T})$*. Then* $\mathbf{Constr}_2(\mathbf{T})$ *is a constructive* $\mathbf{T}$*-system.*

*Proof:* Let $\pi : \Gamma \vdash A$ be a proof in $\mathscr{N}\mathscr{D}_{\mathbf{Int}}$ and suppose that $\Gamma \subseteq \mathbf{Constr}_2(\mathbf{T})$; then, by induction on the structure of $\pi$, one can prove that $A \in \mathbf{Constr}_2(\mathbf{T})$. As a consequence, it holds that $\mathbf{Int} \oplus \mathbf{T} \subseteq \mathbf{Constr}_2(\mathbf{T})$; moreover, since $A \in \mathbf{Constr}_2(\mathbf{T})$ implies $\mathbf{T} \models A$, by the Completeness Theorem of Classical Logic we also have $A \in \mathbf{Cl} \oplus \mathbf{T}$. The constructive properties are immediately satisfied. □

We are not able to state (for any theory $\mathbf{T} \subseteq \mathbf{Constr}_2(\mathbf{T})$) whether $\mathbf{Constr}_2(\mathbf{T})$ is a maximal constructive $\mathbf{T}$-system or not. However, we can prove that, for every $\mathbf{T}$, the principles (Kur), (wGrz), ($\mathrm{KP}_\vee$) and ($\mathrm{KP}_\exists$) of Table 2 belong to $\mathbf{Constr}_2(\mathbf{T})$ (Also (At) belongs to $\mathbf{Constr}_2(\mathbf{T})$). On the other hand, in general (Mk) and the principles (St), (St$\exists$) and (DT) (again, see Table 2) do not belong to $\mathbf{Constr}_2(\mathbf{T})$. The latter fact does not hold if $\mathbf{T}$ is a complete theory with a reachable model, in which case also Grzegorczyk Principle

$$\text{(Grz)} \qquad \forall x(A(x) \vee B) \to \forall x A(x) \vee B \quad \text{with } x \notin \mathrm{FV}(B)$$

belongs both to $\mathbf{Constr}_1(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T})$; we remark, however, that in general (Grz) belongs neither to $\mathbf{Constr}_1(\mathbf{T})$ nor to $\mathbf{Constr}_2(\mathbf{T})$, since, e.g., the addition of (Grz) to Intuitionistic Arithmetic gives rise to Classical Arithmetic [Troelstra, 1973]. Thus, in general one has that $\mathbf{Constr}_1(\mathbf{T}) \not\subseteq \mathbf{Constr}_2(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T}) \not\subseteq \mathbf{Constr}_1(\mathbf{T})$. We point out, on the other hand, that the difference between $\mathbf{Constr}_1(\mathbf{T})$ and $\mathbf{Constr}_2(\mathbf{T})$ collapses for particular theories: for instance, one can prove that, if $\mathbf{T}$ is a classically complete theory with a reachable model, then $\mathbf{Constr}_1(\mathbf{T}) = \mathbf{Constr}_2(\mathbf{T}) = \mathbf{Cl} \oplus \mathbf{T}$.

The following result states sufficient conditions for a theory $\mathbf{T}$ to be contained in $\mathbf{Constr}_2(\mathbf{T})$.

**Theorem 3.9** *Let* $\mathbf{T}$ *be an* $\mathscr{A}$*-theory and let* $H$ *be a closed wff such that* $H \in \mathbf{Cl} \oplus \mathbf{T}$*. Then* $H \in \mathbf{Constr}_2(\mathbf{T})$ *if one of the following conditions holds:*

1. *$H$ is an Harrop-wff;*
2. *$H$ is a $\forall\neg$-wff and $\mathbf{T}$ is atomically complete;*
3. *$H$ is a $\forall\exists$-wff, $\mathbf{T}$ is atomically complete and $\mathbf{T}$ has a reachable model.* □

### 3.3 The PA-systems $\mathbf{Constr}_1(\mathbf{PA})$ and $\mathbf{Constr}_2(\mathbf{PA})$

Let $\mathscr{L}_{\mathbf{PA}}$ be the language of first-order Arithmetic, containing the constant symbol 0, the unary function symbol $S$, the binary function symbols $+$ and $*$ and the binary relation symbol $=$, and let **PA** be the usual axiomatization of Arithmetic. We will denote with **AInt** and **ACl** the **PA**-systems $\mathbf{Int} \oplus \mathbf{PA}$ and $\mathbf{Cl} \oplus \mathbf{PA}$ respectively. Also, $\mathfrak{N}$ will denote the standard (classical) model of **PA**.

According to Theorems 3.7 and 3.9, all the Harrop axioms of **PA** belong to $\mathbf{Constr}_1(\mathbf{PA})$ and to $\mathbf{Constr}_2(\mathbf{PA})$. Since the remaining axioms of **PA** are instances of the Induction Principle (see also §6), which are easily shown to belong both to $\mathbf{Constr}_1(\mathbf{PA})$ and $\mathbf{Constr}_2(\mathbf{PA})$, we get:

**Theorem 3.10** $\mathbf{AInt} \subseteq \mathbf{Constr}_1(\mathbf{PA})$ *and* $\mathbf{AInt} \subseteq \mathbf{Constr}_2(\mathbf{PA})$. □

We already know that $\mathbf{Constr}_1(\mathbf{PA})$ is a maximal semiconstructive **PA**-system, in the next theorem we prove that $\mathbf{Constr}_1(\mathbf{PA})$ is not an effective system.

**Theorem 3.11** $\mathbf{Constr}_1(\mathbf{PA})$ *is not arithmetical.*

*Proof:* We prove that the arithmetical truth can be decided by an oracle for membership in $\mathbf{Constr}_1(\mathbf{PA})$. Let $A$ be any closed wff of $\mathscr{L}_{\mathbf{PA}}$; by a standard effective procedure, we can define a wff $M(x_1, \ldots, x_n)$ in disjunctive normal form such that $Q_1x_1 \ldots Q_nx_nM(x_1, \ldots, x_n)$ is classically equivalent to $A$, where, for $1 \leq j \leq n$, $Q_j$ is one of the quantifiers $\exists, \forall$. By Gödel Incompleteness Theorem, there is a closed wff $G$ such that $G \notin \mathbf{ACl}$ and $\neg G \notin \mathbf{ACl}$; let us define:

$$H(x_1, \ldots, x_n) \equiv (M(x_1, \ldots, x_n) \vee G) \vee \neg(M(x_1, \ldots, x_n) \vee G).$$

We firstly observe that, for every $n$-tuple $\underline{t}$ of closed terms $t_1, \ldots, t_n$, the following fact holds:

(1) $H(\underline{t}) \in \mathbf{Constr}_1(\mathbf{PA})$ implies $\mathfrak{N} \models M(\underline{t})$.

Indeed, suppose that $H(\underline{t}) \in \mathbf{Constr}_1(\mathbf{PA})$; since $\neg(M(\underline{t}) \vee G) \notin \mathbf{Constr}_1(\mathbf{PA})$ (otherwise $\neg G \in \mathbf{ACl}$ would follow) and $G \notin \mathbf{Constr}_1(\mathbf{PA})$, necessarily $M(\underline{t}) \in \mathbf{Constr}_1(\mathbf{PA})$.

Let $H_A$ be the closed wff $Q_1x_1 \ldots Q_nx_nH(x_1, \ldots, x_n)$; by induction on $n$, one can prove that:

(2) $H_A \in \mathbf{Constr}_1(\mathbf{PA})$ iff $\mathfrak{N} \models A$.

Suppose, for instance, that $n = 1$ and $Q_1 = \forall$, that is $A$ is classically equivalent to $\forall x_1M(x_1)$ and $H_A \equiv \forall x_1H(x_1)$. If $\forall x_1H(x_1) \in \mathbf{Constr}_1(\mathbf{PA})$, then $H(t) \in \mathbf{Constr}_1(\mathbf{PA})$ for every closed term $t$. By (1), $\mathfrak{N} \models M(t)$ for every closed term $t$; since $\mathfrak{N}$ is reachable, we can assert that $\mathfrak{N} \models \forall x_1M(x_1)$, hence $\mathfrak{N} \models A$. Conversely, let us assume that $\mathfrak{N} \models A$, that is $\mathfrak{N} \models \forall x_1M(x_1)$, and let $t$ be any closed term. Then $\mathfrak{N} \models M(t)$ and, since $M(t)$ is quantifier free, $M(t) \in \mathbf{AInt}$, which implies $H(t) \in \mathbf{AInt}$. Since $\mathbf{AInt} \subseteq \mathbf{Constr}_1(\mathbf{PA})$, it follows that $H(t) \in \mathbf{Constr}_1(\mathbf{PA})$ for every closed term $t$; moreover, $\forall x_1H(x_1) \in \mathbf{ACl}$. Therefore $\forall x_1H(x_1) \in \mathbf{Constr}_1(\mathbf{PA})$.

By (2), we can conclude that the degree of unsolvability of $\mathbf{Constr}_1(\mathbf{PA})$ is greater than or equal to the non arithmetical set of the closed wff's of $\mathscr{L}_{\mathbf{PA}}$ which are true in the standard model of **PA**. □

As previously said, we do not know whether $\mathbf{Constr}_2(\mathbf{PA})$ is a maximal constructive **PA**-system (hence, we do not know whether it is a maximal semiconstructive **PA**-system). However, following the same idea of the proof of Theorem 3.11 (and using the same wff's $H(x_1, \ldots, x_n)$, even if the argument is slightly more complex) we can state:

**Theorem 3.12** $\mathbf{Constr}_2(\mathbf{PA})$ *is not arithmetical.* □

As previously said, we can define other interesting and very large **PA**-systems (some of which, like $\mathbf{Constr}_1(\mathbf{PA})$, are maximal constructive). However, all these systems turn out to be non arithmetical. We wish to point out that considering Pressburger Arithmetic **PRA** (i.e., the classically complete arithmetical theory of successor and sum, which has a reachable model [Kleene, 1952]) we have that $\mathbf{Constr}_1(\mathbf{PRA}) = \mathbf{Constr}_2(\mathbf{PRA}) = \mathbf{Cl} \oplus \mathbf{PRA}$, thus $\mathbf{Constr}_1(\mathbf{PRA})$ and $\mathbf{Constr}_2(\mathbf{PRA})$ turn out to coincide and to be recursive. Hence, passing from **PRA** to its conservative extension (with respect to classical deduction) **PA**, we pass from recursive sets such as $\mathbf{Constr}_1(\mathbf{PRA})$ to non arithmetical sets such as $\mathbf{Constr}_1(\mathbf{PA})$.

**Remark 3.13** Even if they are quite undecidable (and, moreover, non recursively realizable according to the various notions proposed by the constructive tradition, where also to check recursive realizabilities is undecidable), $\mathbf{Constr}_1(\mathbf{PA})$ and $\mathbf{Constr}_2(\mathbf{PA})$ are rather significant from of the point of view of the Theory of the Recursive Functions. Indeed, every semiconstructive **PA**-system **S** (even if **S**, differently from $\mathbf{Constr}_1(\mathbf{PA})$ and $\mathbf{Constr}_2(\mathbf{PA})$, may be non constructive) has remarkable properties such as the following:

(1) If, for some $A(x_1, \ldots, x_n)$, $\forall x_1 \ldots \forall x_n \exists! y A(x_1, \ldots, x_n, y) \in \mathbf{S}$, then the underlying function from $\mathbf{N}^n$ to $\mathbf{N}$ ($\mathbf{N}$ the set of natural numbers) is recursive;

(2) If, for some $B(x_1, \ldots, x_n)$, $\forall x_1 \ldots \forall x_n (B(x_1, \ldots, x_n) \vee \neg B(x_1, \ldots, x_n)) \in \mathbf{S}$, then the underlying relation on $\mathbf{N}^n$ is recursive.

As a matter of fact, the function and the relation involved in (1) and (2) can be computed by a recursive enumeration of the *recursively enumerable* set of the closed wff's of **ACl**.

Thus, so to say, we may look at $\mathbf{Constr}_1(\mathbf{PA})$ and $\mathbf{Constr}_2(\mathbf{PA})$ as "very great but non effective sets of descriptions of effective functions and relations".

# 4 The information extraction mechanism

In this section we will provide a short presentation of our mechanism to extract information from proofs, giving only the main definitions and results; for a complete discussion and a detailed presentation of all the results we refer the reader to [Ferrari, 1997; Ferrari et al., 1999]. We remark that, even if in this paper all the systems are presented by means of pseudo-natural deduction systems, the extraction mechanism is based on an abstract definition of a calculus allowing to treat also extraction from Gentzen-style, Tableau-style or Hilbert-style calculi.

First of all we define a *proof* on a language $\mathscr{L}_{\mathscr{A}}$ as any finite object $\pi$ such that:

($\pi$1) The (finite) set of wff's of $\mathscr{L}_{\mathscr{A}}$ occurring in $\pi$ is uniquely determined and nonempty;

($\pi$2) The sequent $\Gamma \vdash \Delta$ proved by $\pi$ is uniquely determined, where $\Gamma$ and $\Delta$ are finite sets of wff's of $\mathscr{L}_{\mathscr{A}}$. $\Gamma$ (possibly empty) is the set of *assumptions* of $\pi$ while $\Delta$, which must be nonempty, is the set of *consequences* of $\pi$.

Proofs are characterized by the following attributes: $\mathrm{Seq}(\pi)$ indicates the sequent $\Gamma \vdash \Delta$ proved by $\pi$, $\mathrm{Wffs}(\pi)$ denotes the set of wff's of $\mathscr{L}_{\mathscr{A}}$ occurring in $\pi$, and

$$\mathrm{dg}(\pi) = \max\{\mathrm{dg}(A) : A \in \mathrm{Wffs}(\pi)\}$$

is the *degree of the proof* $\pi$. The compact notation

$$\pi : \Gamma \vdash \Delta$$

will be used to indicate that $\mathrm{Seq}(\pi) = \Gamma \vdash \Delta$.

A *calculus* on $\mathscr{L}_{\mathscr{A}}$ is a pair $\mathbf{C} = (C, [\cdot])$, where $C$ is a recursive set of proofs on the language $\mathscr{L}_{\mathscr{A}}$ and $[\cdot]$ is a recursive map from $C$ into the set of finite subsets of $C$ with the following properties:

($\mathbf{C}1$) $\pi \in [\pi]$;

($\mathbf{C}2$) For every $\pi' \in [\pi]$, $[\pi'] \subseteq [\pi]$;

($\mathbf{C}3$) For every $\pi' \in [\pi]$, $\mathrm{dg}(\pi') \leq \mathrm{dg}(\pi)$.

The map $[\cdot]$ associates with every proof of the calculus the set of its subproofs. We remark that conditions ($\mathbf{C}2$) and ($\mathbf{C}3$) are natural: the former requires that the set of subproofs of a proof also contains the subproofs of its elements; the latter requires that the degree of the subproofs of a proof must not exceed the degree of the proof.

In the following, to simplify the notation, we will identify a calculus $\mathbf{C}$ with the set of its proofs. Now, given a set of proofs $\Pi \subseteq \mathbf{C}$, we denote with $[\Pi]$ the *closure under subproofs* of $\Pi$ in the calculus $\mathbf{C}$. Namely,

$$[\Pi] = \{\pi' : \text{ there exists } \pi \in \Pi \text{ such that } \pi' \in [\pi]\}.$$

In general, $[\Pi]$ is not a recursive set of proofs. If $\Pi$ is finite then, of course, $[\Pi]$ is recursive, and hence $([\Pi], [\cdot]_{\restriction[\Pi]})$ is a calculus, where $[\cdot]_{\restriction[\Pi]}$ is the restriction of $[\cdot]$ to $[\Pi]$.

Given a calculus $\mathbf{C}$, let $\Pi \subseteq \mathbf{C}$; we define the following attributes of $\Pi$:

$\mathrm{Seq}(\Pi)$ : it is the set of all the *sequents proved in* $\Pi$, i.e. $\mathrm{Seq}(\Pi) = \cup_{\pi\in\Pi}\mathrm{Seq}(\pi)$.

$\mathrm{dg}(\Pi)$ : it is the *degree* of $\Pi$, i.e. $\mathrm{dg}(\Pi) = \max\{\mathrm{dg}(\pi) : \pi \in \Pi\}$, where $\mathrm{dg}(\Pi) = \infty$ if $\Pi$ contains proofs of any complexity.

$\mathrm{Theo}(\Pi)$ : it is the set of *theorems proved in* $\Pi$, i.e. $\mathrm{Theo}(\Pi) = \{A : \vdash A \in \mathrm{Seq}(\Pi)\}$.

Given two sets of proofs $\Pi_1$ and $\Pi_2$ on the same language $\mathscr{L}_{\mathscr{A}}$, but possibly belonging to different calculi, we write $\Pi_1 \approx \Pi_2$ iff $\mathrm{Seq}(\Pi_1) = \mathrm{Seq}(\Pi_2)$.

In the following we will be interested in characterizing subsets of a calculus which have some closure properties. To this aim we introduce the notion of generalized rule:

**Definition 4.1 (Generalized rule)** *Given a language $\mathscr{L}_{\mathscr{A}}$, let $\Xi$ be the set of all the sequents in $\mathscr{L}_{\mathscr{A}}$ and let $\Xi^*$ be the set of all the finite sequences of sequents in $\Xi$. A* generalized rule *(on $\mathscr{L}_{\mathscr{A}}$) is a relation $\mathscr{R} \subseteq \Xi^* \times \Xi$.*

We denote with $\epsilon$ the empty sequence of sequents. Let $\sigma^*$ be an element of $\Xi^*$; we will write $\sigma \in \mathscr{R}(\sigma^*)$ as a shorthand for $(\sigma^*, \sigma) \in \mathscr{R}$. The *domain* of $\mathscr{R}$ is the set

$$\mathrm{dom}(\mathscr{R}) = \{\sigma^* \in \Xi^* \ : \ \text{there exists } \sigma \text{ such that } \sigma \in \mathscr{R}(\sigma^*)\}.$$

A set of sequents $\Sigma$ is *$\mathscr{R}$-closed* iff, for every $\sigma, \sigma_1, \ldots, \sigma_n \in \Xi$, if $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$ and $\sigma_1, \ldots, \sigma_n \in \Sigma$ then $\sigma \in \Sigma$. Obviously, a set of proofs (on $\mathscr{L}_{\mathscr{A}}$) is $\mathscr{R}$-closed iff $\mathrm{Seq}(\Pi)$ is $\mathscr{R}$-closed.

Examples of generalized rules we will use in the following are:

*Substitution rule* (Subst).
: The domain of Subst is the set of all the sequents, and, for every substitution $\theta$ of terms for individual variables, $\theta\Gamma \vdash \theta\Delta \in \textsc{Subst}(\Gamma \vdash \Delta)$.

*Intuitionistic Cut rule* (Cut).
: The domain of Cut contains all the sequences of sequents which have the form $\Gamma_1 \vdash H; \Gamma_2, H \vdash A$, and $\Gamma_1, \Gamma_2 \vdash A \in \textsc{Cut}(\Gamma_1 \vdash H; \Gamma_2, H \vdash A)$.

It is immediate to check that any pseudo-natural deduction calculus including $\mathscr{N}\mathscr{D}_{\mathbf{Int}}$ is Cut-closed and Subst-closed.

**Definition 4.2** *Let $\mathscr{R}$ be a generalized rule on $\mathscr{L}_{\mathscr{A}}$ and let $\mathbf{C}$ be a calculus on $\mathscr{L}_{\mathscr{A}}$.*

*(i) A set $\Pi$ of proofs of $\mathbf{C}$ is a* $\mathscr{R}$-subcalculus of $\mathbf{C}$ *if $\Pi$ is $\mathscr{R}$-closed;*

*(ii) A set of proofs $\Pi$ (possibly not belonging to $\mathbf{C}$) is a* generalized $\mathscr{R}$-subcalculus *of* $\mathbf{C}$ *if there is a $\mathscr{R}$-subcalculus $\Pi'$ of $\mathbf{C}$ such that $\Pi \approx \Pi'$.*

The notion of generalized $\mathscr{R}$-subcalculus allows us to work also outside a given calculus $\mathbf{C}$ taking sets of proofs equivalent (in the sense of $\approx$) to $\mathscr{R}$-subcalculi of $\mathbf{C}$. We will be particularly interested in generalized $\mathscr{R}$-subcalculi which are themselves calculi. In this perspective, we introduce the following notion of *abstract calculus* to be subsequently used as the key tool to extract information from proofs.

Let $\mathscr{R}$ be a generalized rule on $\mathscr{L}_{\mathscr{A}}$ and let $\Sigma$ be any set of sequents in the same language. The deductive sequent-system $\mathbb{D}(\mathscr{R}, \Sigma)$ is the set of proof-trees inductively defined as follows:

(i) If $\sigma \in \Sigma$, then $\tau \equiv \sigma$ is a proof-tree of $\mathbb{D}(\mathscr{R}, \Sigma)$ with root $\sigma$ and $\mathrm{depth}(\tau) = 1$.

(ii) If $\tau_1 : \sigma_1, \ldots, \tau_n : \sigma_n$ are proof-trees of $\mathbb{D}(\mathscr{R}, \Sigma)$ (where $\sigma_i$ is the root of $\tau_i$) then, for every $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$, the proof-tree

$$\tau \equiv \cfrac{\tau_1 : \sigma_1 \ \ldots \ \tau_n : \sigma_n}{\sigma}\,\mathscr{R}$$

with root $\sigma$ belongs to $\mathbb{D}(\mathscr{R}, \Sigma)$ and $\mathrm{depth}(\tau) = \max\{\mathrm{depth}(\tau_1), \ldots, \mathrm{depth}(\tau_n)\} + 1$.

We remark that, if both $\mathscr{R}$ and $\Sigma$ are recursive, then $\mathbb{D}(\mathscr{R}, \Sigma)$ is a calculus, where we consider the obvious function $[\cdot]$ determined by the inductive definition of $\mathbb{D}(\mathscr{R}, \Sigma)$.

The calculus $\mathbb{D}(\mathscr{R}, \Sigma)$ allows us to recover the meaning of the generalized rules as inference rules, but abstracting from the particular inference system generating $\Sigma$.

In [Ferrari, 1997; Ferrari et al., 1999] the following important properties of abstract calculi are stated:

**Theorem 4.3** *Let $\mathscr{R}$ be a generalized rule and let* $\mathbf{C}$ *be a $\mathscr{R}$-closed calculus.*

1. *If $\Pi$ is a $\mathscr{R}$-subcalculus of $\mathbf{C}$, then $\Pi \approx \mathbb{D}(\mathscr{R}, \mathrm{Seq}(\Pi))$;*
2. *If $\Pi \subseteq \mathbf{C}$, then $\mathbb{D}(\mathscr{R}, \mathrm{Seq}(\Pi))$ is a generalized $\mathscr{R}$-subcalculus of $\mathbf{C}$.* □

Let $\mathscr{R}$ be a generalized rule on $\mathscr{L}_{\mathscr{A}}$, let $\mathbf{C}$ be a calculus on $\mathscr{L}_{\mathscr{A}}$ and let $\phi$ be a function over natural numbers. $\mathbf{C}$ is *uniformly $\mathscr{R}$-closed (w.r.t. $\phi$)* if $\mathbf{C}$ is $\mathscr{R}$-closed, and, for every $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n \in \mathbf{C}$, if $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$, then there exists a proof $\pi : \sigma \in \mathbf{C}$ with

$$\mathrm{dg}(\pi) \leq \max\left\{\mathrm{dg}(\pi_1), \ldots, \mathrm{dg}(\pi_n), \phi(\mathrm{dg}(\sigma_1)), \ldots, \phi(\mathrm{dg}(\sigma_n)), \phi(\mathrm{dg}(\sigma))\right\}.$$

As an example, the natural deduction calculus $\mathscr{N}\mathscr{D}_{\mathbf{Int}}$ is uniformly Cut-closed (w.r.t. $\phi(x) = x+1$). As a matter of fact, starting from the proofs $\pi_1 : \Gamma \vdash H$ and $\pi_2 : \Delta, H \vdash A$, we can build the proof $\pi : \Gamma, \Delta \vdash A$ as follows:

$$\dfrac{\pi_1 : \Gamma \vdash H \qquad \dfrac{\pi_2 : \Delta, H \vdash A}{\Delta \vdash H \to A}\,\mathrm{I}\!\to}{\Gamma, \Delta \vdash A}\,\mathrm{E}\!\to$$

and $\mathrm{dg}(\pi) = \max\{\mathrm{dg}(\pi_1), \mathrm{dg}(\pi_2), \mathrm{dg}(\Delta, H \vdash A) + 1\}$.

A generalized rule $\mathscr{R}$ is *non-increasing* iff the following conditions hold:

(i) For every $\sigma_1, \ldots, \sigma_n \in \mathrm{dom}(\mathscr{R})$ with $n > 0$, if $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$, then

$$\mathrm{dg}(\sigma) \leq \max\{\mathrm{dg}(\sigma_1), \ldots, \mathrm{dg}(\sigma_n)\};$$

(ii) There exists a positive integer $k$ such that, for every $\sigma \in \mathscr{R}(\epsilon)$, $\mathrm{dg}(\sigma) \leq k$ (we say that $\mathscr{R}$ is *$k$-bounded* if $k$ is the minimum integer for which this condition holds).

The use of non-increasing generalized rules to extract information from sets of proofs with bounded degree (e.g., finite sets of proofs) belonging to some calculus $\mathbf{C}$ guarantees that also the extracted sequents can be proved in $\mathbf{C}$ within a bounded degree, as it is stated by the following theorem:

**Theorem 4.4** *Let $\mathscr{R}$ be a non-increasing (k-bounded) generalized rule on $\mathscr{L}_{\mathscr{A}}$, and let $\mathbf{C}$ be a calculus on $\mathscr{L}_{\mathscr{A}}$ which is uniformly $\mathscr{R}$-closed (w.r.t. some $\phi$). If $\Pi \subseteq \mathbf{C}$ and $\mathrm{dg}(\Pi) \leq k_\Pi$ for some $k_\Pi > 0$, then there exists $h \in \mathbf{N}$ such that, for every $\sigma \in \mathrm{Seq}(\mathbb{D}(\mathscr{R}, \mathrm{Seq}(\Pi)))$, there exists $\pi : \sigma \in \mathbf{C}$ such that $\mathrm{dg}(\pi) \leq h$.* □

Now, we have all the ingredients needed to give the fundamental definitions of *uniformly* constructive and semiconstructive calculus.

**Definition 4.5 (Uniformly constructive calculi)** *Let $\mathbf{C}$ be a calculus on $\mathscr{L}_{\mathscr{A}}$. $\mathbf{C}$ is* uniformly constructive *iff there exists a non-increasing generalized rule $\mathscr{R}$ such that:*

*(i) $\mathbf{C}$ is uniformly $\mathscr{R}$-closed;*

*(ii) For every $\Pi \subseteq \mathbf{C}$, $\mathrm{Theo}(\mathbb{D}(\mathscr{R}, \mathrm{Seq}([\Pi])))$ is constructive.*

**Definition 4.6 (Uniformly semiconstructive calculi)** *Let $\mathbf{C}$ and $\mathbf{C}'$ be two calculi on the same language $\mathscr{L}_{\mathscr{A}}$. $\mathbf{C}$ is* uniformly semiconstructive *in $\mathbf{C}'$ iff there exists a non-increasing generalized rule $\mathscr{R}$ such that:*

*(i)* $\mathbf{C}'$ *is uniformly* $\mathscr{R}$*-closed;*

*(ii)* *For every set* $\Pi$ *of proofs of* $\mathbf{C}$, $\mathrm{Seq}([\Pi]) \subseteq \mathrm{Seq}(\mathbf{C}')$ *and* $\mathrm{Theo}([\Pi])$ *is semiconstructive in* $\mathrm{Theo}(\mathbb{D}(\mathscr{R}, \mathrm{Seq}([\Pi])))$.

## 5 A wide family of uniformly constructive T-systems

In this section we will consider a wide family of theories which are interesting in the area of program synthesis with Abstract Data Types (see, e.g., [Avellone et al., 1999; Miglioli et al., 1994]) and give rise to uniformly constructive calculi. First of all, we will consider arbitrary Harrop theories with a *Generalized Induction Principle*. Then we will study Harrop theories with a *Descending Chain Principle* and Harrop theories with *Markov Principle*. Finally, we will briefly discuss how these results can be extended to **T**-systems including further principles of Table 2.

### 5.1 Harrop Theories with cover set induction

Given a signature $\mathscr{A}$ and an $\mathscr{A}$-theory **T**, we say that **T** *admits a cover set* if there exists a finite set $\mathscr{C}$ of terms of $\mathscr{L}_{\mathscr{A}}$ such that:

(i) No term of $\mathscr{C}$ is a variable;

(ii) For every closed term $t$ of $\mathscr{L}_{\mathscr{A}}$, there is a term $t' \in \mathscr{C}$ such that $\mathbf{T}\vdash_{\overline{\mathbf{Int}}} t = \theta t'$, for some closed substitution $\theta$.

Given a cover set $\mathscr{C} = \{t_1, \ldots, t_n\}$ for **T**, we associate with it the following *Cover Set Induction Rule*:

$$\frac{\Gamma, \Delta_1 \vdash A(t_1) \qquad \ldots \qquad \Gamma, \Delta_n \vdash A(t_n)}{\Gamma \vdash A(x)}\,\mathscr{C}\text{-Ind}$$

where, for $1 \leq i \leq n$, if $t_i$ contains $k_i$ variables $y^i_1, \ldots, y^i_{k_i}$, then $\Delta_i$ is $\{A(y^i_1), \ldots, A(y^i_{k_i})\}$; if $k_i = 0$ then $\Delta_i$ is empty. The variables $y^i_1, \ldots, y^i_{k_i}$ are called *proper parameters* of the Cover Set Induction Rule and must not occur free in $\Gamma, A(x)$. The wff's in $\Delta_i$ are the induction hypotheses. We extend to the rule $\mathscr{C}$-Ind Conditions (P1) and (P2) on proper parameters made in §2.

Now, let us denote with $\mathscr{ND}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{T})$ the calculus obtained by adding the rule $\mathscr{C}$-Ind to $\mathscr{ND}_{\mathbf{Int}}(\mathbf{T})$. Given a theory **T**, we say that $\mathscr{ND}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{T})$ *contains the rule* $\mathscr{C}$-Ind *in the adequate context* if $\mathscr{C}$ is a cover set for **T**.

In the following steps, we will prove that, if **Hr** is an Harrop-Theory and $\mathscr{ND}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ contains $\mathscr{C}$-Ind in the adequate context, then $\mathscr{ND}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ is uniformly constructive (of course, in these hypotheses, one almost immediately shows that, for every sequent $\vdash A$ provable in $\mathscr{ND}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$, $A \in \mathbf{Constr}_1(\mathbf{Hr})$ and $A \in \mathbf{Constr}_2(\mathbf{Hr})$). To this aim, we will use the generalized rule RHR consisting of the union of the generalized rules CUT, SUBST and of the generalized rules of Table 3. Since any of the generalized rules giving rise to RHR is non increasing and uniformly translatable into the calculus $\mathscr{ND}_{\mathbf{Int}}$, also RHR meets these properties.

To get the main result we need the following notion of evaluation.

**Definition 5.1 (Closed evaluation)** *Let* $\Pi$ *be a set of proofs on* $\mathscr{L}_{\mathscr{A}}$ *and let* $A \in \mathscr{L}_{\mathscr{A}}$. $A$ is evaluated in $\Pi$ *iff the following conditions hold:*

| | | | |
|---|---|---|---|
| $\text{R}_{\text{E}\wedge}$ : | $\vdash A$ | $\in$ | $\text{R}_{\text{E}\wedge}(\vdash A \wedge B)$ |
| | $\vdash B$ | $\in$ | $\text{R}_{\text{E}\wedge}(\vdash A \wedge B)$ |
| $\text{R}_{\text{E}\forall}$ : | $\vdash A(x)$ | $\in$ | $\text{R}_{\text{E}\forall}(\vdash \forall x A(x))$ |
| $\text{R}_{\text{MP}}$ : | $\vdash B$ | $\in$ | $\text{R}_{\text{MP}}(\vdash A \to B; \vdash A)$ |
| $\text{I}_{\text{D}_1}$ : | $\vdash x = x$ | $\in$ | $\text{I}_{\text{D}_1}(\epsilon)$ |
| $\text{I}_{\text{D}_2}$ : | $\Gamma, \Delta \vdash A(t')$ | $\in$ | $\text{I}_{\text{D}_2}(\Gamma \vdash A(t); \Delta \vdash t = t')$ |

Table 3: The generalized rule RHR

*(i) There is a proof* $\pi : \vdash A \in \Pi$*;*

*(ii) For every closed instance* $\theta A$ *of* $A$*, one of the following conditions holds:*

- *(a)* $\theta A$ *is atomic or negated;*
- *(b)* $\theta A \equiv B \wedge C$*, and both* $B$ *and* $C$ *are evaluated in* $\Pi$*;*
- *(c)* $\theta A \equiv B \vee C$*, and either* $B$ *is evaluated in* $\Pi$ *or* $C$ *is evaluated in* $\Pi$*;*
- *(d)* $\theta A \equiv B \to C$*, and either* $B$ *is not evaluated in* $\Pi$ *or* $C$ *is evaluated in* $\Pi$*;*
- *(e)* $\theta A \equiv \exists x B(x)$*, and* $B(t/x)$ *is evaluated in* $\Pi$ *for some closed term* $t$ *of* $\mathscr{L}_{\mathscr{A}}$*;*
- *(f)* $\theta A \equiv \forall x B(x)$*, and, for every closed term* $t$ *of* $\mathscr{L}_{\mathscr{A}}$*,* $B(t/x)$ *is evaluated in* $\Pi$*.*

A set $\Gamma$ of wff's is evaluated in a set of proofs $\Pi$ if every wff $A \in \Gamma$ is evaluated in $\Pi$.

First of all we prove a general result about provability in $\mathbb{D}(\mathscr{R}, \Sigma)$.

**Lemma 5.2** *Let* $\mathscr{R}$ *be any generalized rule including* CUT *and* SUBST *and let* $\mathbf{C}$ *be an* $\mathscr{R}$*-closed calculus. For every set of proofs* $\Pi \subseteq \mathbf{C}$ *and every proof* $\pi : \Gamma \vdash A$ *belonging to the closure under substitution of* $[\Pi]$*, if* $\Gamma$ *is evaluated in* $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$*, then the sequent* $\vdash A$ *is provable in* $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$*.*

*Proof:* Since $\pi$ belongs to the closure under substitution of $[\Pi]$, there exist $\pi' : \Gamma' \vdash A' \in [\Pi]$ and a substitution $\theta$ of individual variables such that $\theta\Gamma' \vdash \theta A' \equiv \Gamma \vdash A$. Thus, by definition, $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$ contains a proof of the sequent $\Gamma' \vdash A'$ and hence, since $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$ is SUBST-closed, there exists a proof $\tau' : \Gamma \vdash A$ in $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$. Moreover, since $\Gamma = \{H_1, \ldots, H_n\}$ is evaluated in $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$, there exist proofs $\tau_1 : \vdash H_1, \ldots, \tau_n : \vdash H_n$ in $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$. Since $\mathbb{D}(\mathscr{R}, \text{Seq}([\Pi]))$ is CUT-closed and contains the proofs $\tau'$, $\tau_1, \ldots \tau_n$, it also contains a proof $\tau^* : \vdash A$. □

Now, let $\mathbb{D}_{\text{RHR}}(\Pi)$ be an abbreviation for $\mathbb{D}(\text{RHR}, \text{Seq}(\Pi))$.

**Lemma 5.3** *Let* $\mathbf{Hr}$ *be an Harrop theory such that* $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ *contains* $\mathscr{C}$-Ind *in the adequate context and let* $\Pi$ *be any set of proofs of* $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$*. For every proof* $\pi : \Gamma \vdash A$ *belonging to the closure under substitution of* $[\Pi]$*, if* $\Gamma$ *is evaluated in* $\mathbb{D}_{\text{RHR}}([\Pi])$*, then* $A$ *is evaluated in* $\mathbb{D}_{\text{RHR}}([\Pi])$*.*

*Proof:* Point (i) of Definition 5.1 follows from Lemma 5.2; to prove Point (ii) we proceed by induction on depth($\pi$).

*Basis*: If depth($\pi$) $= 0$, we have two cases: the only rule applied in $\pi$ is either an assumption introduction or an axiom-rule. In the former case $\Gamma = \{A\}$ and hence $A$ is trivially evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. In the latter case $A$ is a closed Harrop wff, $\Gamma$ is empty and we can easily prove, by a secondary induction on the degree of the Harrop wff $A$, that $\vdash A$ provable in $\mathbb{D}_{\text{RHR}}([\Pi])$ implies $A$ evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. As an example, let us consider the case $A \equiv B \to C$. Let us suppose that $B$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$; then there exists a proof $\tau' : \vdash B$ in $\mathbb{D}_{\text{RHR}}([\Pi])$, and since $\mathbb{D}_{\text{RHR}}([\Pi])$ is RMP-closed and contains the proof of the sequent $\vdash B \to C$ (by Point (i)), it also contains a proof of $\vdash C$, which is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$ by induction hypothesis.

*Step*: Let us suppose that depth($\pi$) $= h+1$. The proof goes on by cases according to the last rule applied in $\pi$; here we only discuss two representative cases.

*Disjunction Elimination.*

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_0 : \Gamma_0 \vdash B_1 \vee B_2 \qquad \pi_1 : \Gamma_1, B_1 \vdash A \qquad \pi_2 : \Gamma_2, B_2 \vdash A}{\Gamma \vdash A}\,\mathrm{E}\vee$$

Let $\theta A$ be any closed instance of $A$. Since $\theta\Gamma_0$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$, $\theta\pi_0$ belongs to the closure under substitution of $[\Pi]$, and depth($\pi_0$) $\leq h$, we get, by induction hypothesis, that $\theta B_1 \vee \theta B_2$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. Thus, there exists $i \in \{1, 2\}$ such that $\theta B_i$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$ and, since $\theta\pi_i : \theta\Gamma_i, \theta B_i \vdash \theta A$ belongs to the closure under substitution of $[\Pi]$, we get, by induction hypothesis, that $\theta A$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$.

*Generalized induction rule*:

$$\pi : \Gamma \vdash A \equiv \cfrac{\pi_1 : \Gamma, \Delta_1 \vdash B(t_1) \qquad \ldots \qquad \pi_n : \Gamma, \Delta_n \vdash B(t_n)}{\Gamma \vdash B(x)}\,\mathscr{C}\text{-Ind}$$

Let us consider an arbitrary closed instance $\theta B(x)$ of $B(x)$ and let $t = \theta(x)$. The proof goes on by a secondary induction on the structure of the term $t$. The basis is the case where $t \equiv c$ is a constant symbol of $\mathscr{A}$. In this case $c$ belongs to the cover set $\mathscr{C}$, then, by the definition of the rule $\mathscr{C}$-Ind, there exists a subproof $\pi_i : \Gamma \vdash B(c)$, for some $i \in \{1, \ldots, n\}$; thus, the assertion immediately follows from the principal induction hypothesis applied to the proof $\theta\pi_i$. Now, let us suppose that the assertion holds for any term $t'$ with complexity less than or equal to $j$, and let $j+1$ be the complexity of $t$. By the definition of a cover set, there exists a term $t_i \in \mathscr{C}$ such that, for some substitution $\theta'$ only acting on the variables $y^i_1, \ldots, y^i_{k_i}$ , $t = \theta' t_i \in \mathbf{Int} \oplus \mathbf{Hr}$. Let us consider the proof $\pi_i : \Gamma, \Delta_i \vdash B(t_i)$, and let us apply the substitution $\theta'$ to this proof. By the convention on the proper parameters, we have that the proof $\theta\theta'\pi_i$ is a proof of the sequent $\theta\Gamma, \theta\theta'\Delta_i \vdash \theta B(\theta' t_i)$. Notice that $\theta\theta'\Delta_i$ contains wff's of the kind $\theta B(t')$, with $t'$ a closed term with complexity less than or equal to $j$. Thus, we can apply the secondary induction hypothesis and deduce that $\theta'\Delta_i$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$; finally, by applying the principal induction hypothesis to the proof $\theta\theta'\pi_i$, we get that $\theta B(\theta' t_i)$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. Now, it is easy to prove, by induction on the term $t$ and using the assumption $t = \theta' t_i \in \mathbf{Int} \oplus \mathbf{Hr}$, that $\theta B(t)$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$ iff $\theta B(\theta' t_i)$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. □

**Corollary 5.4** *Let* **Hr** *be an Harrop theory such that* $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ *contains* $\mathscr{C}$-Ind *in the adequate context and let* $\Pi$ *be a set of proofs of* $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$. *For every* $\tau : \Gamma \vdash A \in \mathbb{D}_{\text{RHR}}([\Pi])$ *and every substitution* $\theta$*, if* $\theta\Gamma$ *is evaluated in* $\mathbb{D}_{\text{RHR}}([\Pi])$*, then* $\theta A$ *is evaluated in* $\mathbb{D}_{\text{RHR}}([\Pi])$.

*Proof:* Point (i) of Definition 5.1 immediately follows from Lemma 5.2, taking as **C** the calculus $\mathbb{D}_{\text{RHR}}([\Pi])$.

To prove Point (ii), we proceed by induction on the overall number of applications of the generalized rules Cut, Re∧, Re∀, Rmp, Id$_1$ and Id$_2$ occurring in $\tau$.

*Basis*: If none of these rules is applied in $\tau$, then $\tau : \Gamma \vdash A$ is obtained by applying a (possibly empty) sequence of Subst to a sequent in Seq($[\Pi]$). Hence, there exists a proof $\pi' : \Gamma' \vdash A'$ in $[\Pi]$ such that $\theta'\Gamma' \vdash \theta' A' \equiv \Gamma \vdash A$ for some substitution $\theta'$; then, also the sequent $\theta\Gamma \vdash \theta A \equiv \theta\theta'\Gamma' \vdash \theta\theta' A'$ has a proof in the closure under substitution of $[\Pi]$, and, since $\theta\Gamma$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$, by Lemma 5.3, $\theta A$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$.

*Step*: Now, let us suppose that $\tau$ contains $h+1$ applications of the above rules. The proof goes on according to the last among these rules applied in $\tau$. As an example we treat the case of the Cut rule.

*Rule* Cut. Then the proof $\tau : \Gamma \vdash A$ has the following form:

$$\cfrac{\cfrac{\cfrac{\tau_1 : \Gamma'_1 \vdash H \qquad \tau_2 : \Gamma'_2, H \vdash A}{\Gamma' \vdash A'}\,\textsc{Cut}}{\vdots}\,\textsc{Subst}}{\theta'\Gamma' \vdash \theta' A'}\,\textsc{Subst}$$

where $\Gamma' = \Gamma'_1 \cup \Gamma'_2$ and $\tau$ ends with a (possibly empty) sequence of applications of Subst. Since $\theta\theta'\Gamma_1 \subseteq \theta\theta'\Gamma'$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$, by induction hypothesis on the proof

$$\cfrac{\tau_1 : \Gamma'_1 \vdash H}{\theta\theta'\Gamma'_1 \vdash \theta\theta' H}\,\textsc{Subst}$$

we get that $\theta\theta' H$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. Hence, $\theta\theta'\Gamma'_2, \theta\theta' H$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$, and, by induction hypothesis on the proof

$$\cfrac{\tau_2 : \Gamma'_2, H \vdash A'}{\theta\theta'\Gamma'_2, \theta\theta' H \vdash \theta\theta' A'}\,\textsc{Subst}$$

$\theta\theta' A' \equiv \theta A$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$.

□

Now, if **Hr** is an Harrop theory such that $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ contains $\mathscr{C}$-Ind in the adequate context and $\Pi$ is any set of proofs of $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$, then it is immediate to check that the set of theorems of $\mathbb{D}_{\text{RHR}}([\Pi])$ is constructive. Let us suppose, as an example, that $\exists x A(x)$ is a closed wff belonging to Theo($\mathbb{D}_{\text{RHR}}([\Pi])$). Then there exists a proof $\tau : \vdash \exists x A(x)$ in $\mathbb{D}_{\text{RHR}}([\Pi])$. Since the empty set of premises is trivially evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$, by Corollary 5.4 we have that $\exists x A(x)$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. By Definition 5.1, it follows that there exists a closed term $t$ of $\mathscr{L}_{\mathscr{A}}$ such that $A(t)$ is evaluated in $\mathbb{D}_{\text{RHR}}([\Pi])$. Hence, by Point (i) of Definition 5.1, $A(t) \in$ Theo($\mathbb{D}_{\text{RHR}}([\Pi])$) and hence Theo($\mathbb{D}_{\text{RHR}}([\Pi])$) satisfies (Ed). In a similar way we can prove that Theo($\mathbb{D}_{\text{RHR}}([\Pi])$) enjoys the disjunction property. Thus, by the constructivity of Theo($\mathbb{D}_{\text{RHR}}([\Pi])$), recalling that RHR is a non-increasing generalized rule and $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ is uniformly RHR-closed, we deduce:

**Theorem 5.5** *Let* **Hr** *be an Harrop theory such that* $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ *contains* $\mathscr{C}$-Ind *in the adequate context. Then the calculus* $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ *is uniformly constructive.* □

## 5.2 Harrop Theories with Descending Chain Principle

Let us consider a signature $\mathscr{A}$ containing the binary relation $<$ and let **T** be an $\mathscr{A}$-theory axiomatizing $<$ as an irreflexive and transitive relation (e.g., **T** may contain the Harrop axioms $\forall x(\neg x < x)$ and $\forall x \forall y \forall z(x < y \wedge y < z \to x < z)$). We call *Descending Chain Principle* the principle

$$\text{(DCP)} \qquad \exists x A(x) \wedge \forall y(A(y) \to \exists z((A(z) \wedge z < y) \vee B)) \to B \quad \text{with } y, z \notin \mathrm{FV}(B).$$

For a discussion on the meaning, power, and relevance for Computer Science of (DCP) we refer the reader to [Ferrari, 1997; Ferrari et al., 1999; Miglioli et al., 1994; Miglioli and Ornaghi, 1981].

The formulation of (DCP) as a pseudo-natural deduction rule can be given as follows:

$$\frac{\Gamma \vdash \exists x A(x) \qquad \Gamma, A(y) \vdash \exists z(A(z) \wedge z < y) \vee B}{\Gamma \vdash B}\,\text{DCP}$$

where $y$ is the *proper parameter* of DCP and $y \notin \mathrm{FV}(\Gamma)$ and $y \notin \mathrm{FV}(B)$.

Let $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{T})$ be the pseudo-natural deduction calculus over the language $\mathscr{L}_{\mathscr{A}}$ obtained by adding to $\mathscr{N}\mathscr{D}_{\mathbf{Int}}(\mathbf{T})$ the rule DCP. Now, we say that the calculus $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{T})$ *contains the rule* DCP *in the adequate context* if there exists an $\mathscr{A}$-structure $\mathfrak{M}$ such that:

(dcp-1) $\mathfrak{M} \models \mathbf{T} \oplus \text{(DCP)}$;

(dcp-2) The relation $<^{\mathfrak{M}}$ (that is, the interpretation in the structure $\mathfrak{M}$ of the relation symbol $<$) is *well founded.*

Now, under the previous hypotheses, one can show that, for every sequent $\vdash A$ provable in $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{Hr})$, $A \in \mathbf{Constr}_1(\mathbf{Hr})$ and $A \in \mathbf{Constr}_2(\mathbf{Hr})$. Also, given an Harrop theory **Hr**, the proof of uniform constructivity of $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{Hr})$ is similar to the one given in the previous section for the calculus $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ and uses the same generalized rule RHR. Here we only discuss the main difference, that is we analyze the rule DCP in the frame of the proof of the induction step of Lemma 5.3.

Let **Hr** be an Harrop theory such that $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{Hr})$ contains the rule DCP in the adequate context. Let $\Pi \subseteq \mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{Hr})$, let $\pi : \Gamma \vdash B$ be a proof belonging to the closure under substitution of $[\Pi]$ such that $\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$ and let $\pi : \Gamma \vdash B$ have the following form:

$$\frac{\pi_1 : \Gamma \vdash \exists x A(x) \qquad \pi_2 : \Gamma, A(y) \vdash \exists x(A(x) \wedge x < y) \vee B}{\Gamma \vdash B}\,\text{DCP}$$

Let us suppose that some closed instance $\theta B$ of $B$ is not evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. By induction hypothesis, $\theta \exists x A(x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, and hence there exists a closed term $t_0$ such that $\theta A(t_0/x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. By the conventions on the proper parameters, we have that $\pi_2[t_0/y]$ is a proof of the sequent $\theta\Gamma, \theta A(t_0/y) \vdash \theta\exists x(A(x) \wedge x < t_0) \vee \theta B$. Thus, by induction hypothesis, $\theta\exists x(A(x) \wedge x < t_0) \vee \theta B$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Since $\theta B$ is not evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, this means that there exists a closed term $t_1$ such that both $\theta A(t_1)$ and $t_1 < t_0$ are evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Now, iterating this argument, we can find an infinite sequence $t_0, t_1, \ldots, t_n, \ldots$ of closed terms of $\mathscr{L}_{\mathscr{A}}$ such that

$$t_1 < t_0 \ , \ t_2 < t_1 \ , \ldots , \ t_{n+1} < t_n \ , \ldots$$

are evaluated, and hence provable, in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Since $\mathscr{N\!D}_{\mathbf{DCP}}(\mathbf{Hr})$ is RHR closed, this implies that all these wff's are provable in $\mathscr{N\!D}_{\mathbf{DCP}}(\mathbf{Hr})$, and hence they are provable from $\mathbf{Hr} \cup$ (DCP) using Classical Logic. This implies that, in every classical model $\mathfrak{M}$ of $\mathbf{Hr} \oplus$ (DCP), the relation $<^{\mathfrak{M}}$ contains an infinite descending chain; but this contradicts Conditions (dcp-1) and (dcp-2), hence $B$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

Thus, we can assert:

**Theorem 5.6** *Let* **Hr** *be an Harrop theory such that* $\mathscr{N\!D}_{\mathbf{DCP}}(\mathbf{Hr})$ *contains* DCP *in the adequate context. Then the calculus* $\mathscr{N\!D}_{\mathbf{DCP}}(\mathbf{Hr})$ *is uniformly constructive.* □

## 5.3 Harrop Theories with Markov Principle

To provide a further example of a uniformly constructive **T**-system, we consider the well known *Markov Principle* (Mk) given in Table 2. Detailed discussions about the relevance of this principle in the area of constructivism and for program synthesis can be found in [Miglioli and Ornaghi, 1981; Troelstra, 1973; Voronkov, 1987]. (We note that (Mk) is recursively realizable in the sense of Kleene [Kleene, 1952], as stated in [Troelstra, 1973].) The formulation of (Mk) as a pseudo-natural deduction rule can be given as follows:

$$\frac{\Gamma, \neg\neg\exists x A(z) \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma, \neg\neg\exists x A(x) \vdash \exists x A(x)}\,\text{Mk}$$

Let $\mathscr{N\!D}_{\mathbf{Mk}}(\mathbf{T})$ be the pseudo-natural deduction calculus over the language $\mathscr{L}_{\mathscr{A}}$ obtained by adding to $\mathscr{N\!D}_{\mathbf{Int}}(\mathbf{T})$ the rule Mk. We say that the calculus $\mathscr{N\!D}_{\mathbf{Mk}}(\mathbf{T})$ *contains the rule* Mk *in the adequate context* if:

(mk) There exists an $\mathscr{A}$-structure $\mathfrak{M}$ such that $\mathfrak{M} \models \mathbf{T}$ and $\mathfrak{M}$ is reachable.

Also in this case, for an Harrop theory **Hr**, we do not provide the complete proof of uniform constructivity, but we analyze it as a case of the proof of Lemma 5.3, corresponding to the application of the rule Mk. Here, $\pi : \Gamma \vdash H$ has the following form:

$$\frac{\pi_1 : \Gamma', \neg\neg\exists x A(x) \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma', \neg\neg\exists x A(x) \vdash \exists x A(x)}\,\text{Mk}$$

We must show that, if $\Gamma', \neg\neg\exists x A(x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$, then $\exists x A(x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Let us consider a closed instance $\theta\exists x A(x)$ of $\exists x A(x)$. By induction hypothesis on the proof $\theta\pi_1$, we have that $\theta\forall x(A(x) \vee \neg A(x))$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Hence, by definition, for every closed term $t$ of $\mathscr{L}_{\mathscr{A}}$, $\theta A(t/x) \vee \neg\theta A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Let us suppose that, for every closed term $t$, $\neg\theta A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. This implies that, for each term $t$, there exists a proof $\tau_t : \vdash \neg\theta A(t/x)$ in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. Since $\mathscr{N}\mathscr{D}_{\mathbf{Mk}}(\mathbf{Hr})$ is RHR-closed and $\mathbf{Int} \oplus (\mathrm{Mk}) \oplus \mathbf{Hr} \subseteq \mathbf{Cl} \oplus \mathbf{Hr}$, by the above facts, we deduce:

(1) $\theta\exists x A(x) \in \mathbf{Cl} \oplus \mathbf{Hr}$

(2) For every closed term $t$ of $\mathscr{L}_{\mathscr{A}}$, $\theta\neg A(t/x) \in \mathbf{Cl} \oplus \mathbf{Hr}$.

But, by the Soundness Theorem of Classical Logic, it is easy to check that Facts (1) and (2) and Condition (mk) lead to a contradiction. Hence, there must exist at least a closed term $t$ such that $\theta A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$. This immediately implies that $\theta\exists x A(x)$ is evaluated in $\mathbb{D}_{\mathbf{Hr}}([\Pi])$.

This should convince the reader that, arguing along the lines of §5.1, one can prove the following result:

**Theorem 5.7** *Let* **Hr** *be an Harrop theory such that* $\mathscr{N}\mathscr{D}_{\mathbf{Mk}}(\mathbf{Hr})$ *contains* Mk *in the adequate context. Then, the calculus* $\mathscr{N}\mathscr{D}_{\mathbf{Mk}}(\mathbf{Hr})$ *is uniformly constructive.* □

## 5.4 Further uniformly constructive calculi

Of course, we can combine calculi such as the ones considered in §5.1, 5.2 and 5.3 into single bigger ones and show (without affecting the involved generalized rules) that the resulting calculi are still uniformly constructive.

To be more precise, let $<$ be a binary relation symbol belonging to a signature $\mathscr{A}$, and let **Hr** be an Harrop theory in the language $\mathscr{L}_{\mathscr{A}}$ axiomatizing $<$ as an irreflexive and transitive relation. Let $\mathscr{C}$ be a cover set for **Hr**. Suppose **Hr** to simultaneously satisfy the following conditions:

(i) The calculus $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$ contains the rule $\mathscr{C}$-Ind in the adequate context;

(ii) The calculus $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{Hr})$ (related to the relation symbol $<$) contains the rule DCP in the adequate context;

(iii) The calculus $\mathscr{N}\mathscr{D}_{\mathbf{Mk}}(\mathbf{Hr})$ contains the rule Mk in the adequate context.

Let $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind},\mathbf{DCP},\mathbf{Mk}}(\mathbf{Hr})$ be the pseudo-natural deduction calculus containing all the rules of $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind}}(\mathbf{Hr})$, $\mathscr{N}\mathscr{D}_{\mathbf{DCP}}(\mathbf{Hr})$ and $\mathscr{N}\mathscr{D}_{\mathbf{Mk}}(\mathbf{Hr})$. Then we have:

(F1) $\mathscr{N}\mathscr{D}_{\mathscr{C}\text{-}\mathbf{Ind},\mathbf{DCP},\mathbf{Mk}}(\mathbf{Hr})$ is uniformly constructive.

Let us remark that (F1) involves a wide family of theories. In particular, the theory **PA** of Arithmetic can be expressed by a set of Harrop axioms enriched by an Induction Rule meeting the conditions of (F1); further, **PA** fulfills the requirements of (F1) also with respect to Markov Principle, since its standard model is reachable; finally, in the frame of **PA** one can define various irreflexive and transitive relations making adequate the corresponding Descending Chain Principles (it is to be pointed out, however, that taking $<$ as the usual strict order on the natural numbers, the corresponding Descending Chain Principle is already derivable in **AInt**).

A further extension can be obtained by adding to the calculi considered in (F1) other principles of $\mathbf{Constr}_1(\mathbf{Hr})$ explained in Table 2. In this sense, Kuroda Principle (Kur) does not give rise to new difficulties and its addition provides uniformly constructive calculi which can be handled by means of the previous generalized rules. But we can do more, i.e., we can add to any calculus considered in (F1) (possibly enriched with (Kur)) also the Extended Scott Principle (St∃) (or the weaker Scott Principle (St)) without affecting their uniform constructivity (the proof of this fact, which requires the introduction of new generalized rules and a suitable generalization of the notion of evaluation, is omitted; for more information, the reader is referred to [Ferrari et al., 1999]).

Thus, indicating with $\mathscr{N}\mathscr{D}_{\mathscr{C}\textbf{-Ind},\textbf{DCP},\textbf{Mk},\textbf{Kur},\textbf{St}\exists}(\textbf{Hr})$ the extension of the calculus of (F1) with inference rules for (Kur) and (St∃), under the adequacy hypotheses involved in (F1) we get:

(F2) $\mathscr{N}\mathscr{D}_{\mathscr{C}\textbf{-Ind},\textbf{DCP},\textbf{Mk},\textbf{Kur},\textbf{St}\exists}(\textbf{Hr})$ is uniformly constructive.

One might add to any calculus of (F2) further principles of $\textbf{Constr}_1(\textbf{Hr})$ explained in Table 2, but the addition of (wGrz) and/or (DT) gives rise to (yet uniformly semiconstructive) calculi which cannot be extended into constructive ones without affecting their effectiveness, in line with the results explained in the next section. On the other hand, in general (i.e., for every Harrop theory **Hr**) we cannot extend (F2) to calculi containing principles of Table 2 such as ($KP_\vee$) and ($KP_\exists$) (which, incidentally, admit suitable recursive realizability interpretations, as explained in [Troelstra, 1973]), since the following *constructive incompatibility* phenomena are well known or can be easily proved:

(A) The addition of both (Mk) and ($KP_\exists$) to **AInt** gives rise to **ACl** [Troelstra, 1973];

(B) The addition of both (Mk) and ($KP_\vee$) to **AInt** gives rise to a **PA**-system which is not semiconstructive;

(C) The addition of both (St∃) and ($KP_\exists$) to **AInt** gives rise to a **PA**-system which is not semiconstructive.

However, denoting with $\mathscr{N}\mathscr{D}_{\mathscr{C}\textbf{-Ind},\textbf{DCP},\textbf{KP}_\vee,\textbf{KP}_\exists,\textbf{Kur}}(\textbf{Hr})$ the calculus obtained by adding rules for (DCP), ($KP_\vee$), ($KP_\exists$) and (Kur) to the calculus $\mathscr{N}\mathscr{D}_{\mathscr{C}\textbf{-Ind}}(\textbf{Hr})$, assuming that the adequacy conditions involved in (F1) and (F2) are satisfied, and using appropriate generalized rules, one can prove:

(F3) $\mathscr{N}\mathscr{D}_{\mathscr{C}\textbf{-Ind},\textbf{DCP},\textbf{KP}_\vee,\textbf{KP}_\exists,\textbf{Kur}}(\textbf{Hr})$ is uniformly constructive.

The further extension of any calculus of (F3) by means of the principle (wGrz) gives rise to uniformly semiconstructive calculi such as the ones considered in the next section.

## 6 Uniformly semiconstructive PA-systems

In this section we will investigate the uniform semiconstructivity of some calculi including Intuitionistic Arithmetic (but we might as well consider a more general family of calculi involving arbitrary Harrop theories, Cover Set Inductions and Descending Chain Principles, as made in the previous section). For the sake of simplicity, instead of considering the calculi $\mathscr{N}\mathscr{D}_{\textbf{Int}}(\textbf{PA})$ and $\mathscr{N}\mathscr{D}_{\textbf{Cl}}(\textbf{PA})$, we will consider the more usual calculi $\mathscr{N}\mathscr{D}_{\textbf{AInt}}$ and $\mathscr{N}\mathscr{D}_{\textbf{ACl}}$, obtained by adding the rules of Table 4 to $\mathscr{N}\mathscr{D}_{\textbf{Int}}$ and $\mathscr{N}\mathscr{D}_{\textbf{Cl}}$ respectively.

$$\frac{\Gamma \vdash 0 = S(x)}{\Gamma \vdash \bot}\,\mathrm{S}_1 \qquad \frac{\Gamma \vdash S(t) = S(t')}{\Gamma \vdash t = t'}\,\mathrm{S}_2$$

$$\frac{}{\Gamma \vdash t + 0 = t}\,+_1 \qquad \frac{}{\Gamma \vdash t + S(t') = S(t + t')}\,+_2$$

$$\frac{}{\Gamma \vdash t * 0 = 0}\,*_1 \qquad \frac{}{\Gamma \vdash t * S(t') = t * t' + t}\,*_2$$

$$\frac{\Gamma \vdash A(0) \quad \Delta, A(p) \vdash A(S(p))}{\Gamma, \Delta \vdash A(x)}\,\mathrm{Ind} \quad \text{where } p \text{ does not occur free in } \Delta$$

Table 4: Rules for $\mathscr{N}\mathscr{D}_{\mathbf{AInt}}$

## 6.1 A uniformly semiconstructive PA-system included in $\mathbf{Constr}_2(\mathbf{PA})$

In this section we will discuss the uniformly semiconstructive system $\mathbf{AInt}^+$ obtained by adding to Intuitionistic Arithmetic **AInt** the axioms (Kur), (wGrz), ($\mathrm{KP}_\vee$) and ($\mathrm{KP}_\exists$) of Table 2, where (wGrz) is a weakened form of the well known *Grzegorczyk Principle* (Grz), ($\mathrm{KP}_\vee$) is *Kreisel-Putnam Principle*, a principle well known in the literature of propositional intermediate logics (see, e.g., [Kreisel and Putnam, 1957; Troelstra, 1973]) and ($\mathrm{KP}_\exists$), which is also known in the area of constructivism as (IP) [Troelstra, 1973], naturally completes the meaning of ($\mathrm{KP}_\vee$) at the predicate level. A maximal constructive intermediate predicate logic including ($\mathrm{KP}_\vee$) and ($\mathrm{KP}_\exists$) is studied in [Avellone et al., 1996].

The above principles can be expressed by the pseudo-natural deduction rules of Table 5; in the following we will prove that the calculus $\mathscr{N}\mathscr{D}_{\mathbf{AInt}^+}$ obtained by adding the rules of Table 5 to $\mathscr{N}\mathscr{D}_{\mathbf{AInt}}$ is uniformly semiconstructive.

We denote with RAINT+ the union of the generalized rules of Table 6 and of the generalized rules CUT and SUBST. It is easy to check that RAINT+ is non-increasing and that the calculus $\mathscr{N}\mathscr{D}_{\mathbf{ACl}}$ is uniformly RAINT+-closed.

$$\frac{\Gamma_1 \vdash \forall x \neg\neg A(x) \quad \Gamma_2 \vdash \forall x (A(x) \vee B)}{\Gamma_1, \Gamma_2 \vdash \forall x A(x) \vee B}\,\mathrm{wGrz} \quad \text{with } x \notin \mathrm{FV}(B)$$

$$\frac{\Gamma \vdash \forall x \neg\neg A(x)}{\Gamma \vdash \neg\neg \forall x A(x)}\,\mathrm{Kur} \qquad \frac{\Gamma, \neg A \vdash B \vee C}{\Gamma \vdash (\neg A \to B) \vee (\neg A \to C)}\,\mathrm{KP}_\vee \qquad \frac{\Gamma, \neg A \vdash \exists x B(x)}{\Gamma \vdash \exists x (\neg A \to B(x))}\,\mathrm{KP}_\exists$$

Table 5: Rules for $\mathscr{N}\mathscr{D}_{\mathbf{AInt}^+}$

Now, we will prove that, for every set $\Pi$ of proofs of $\mathscr{N}\mathscr{D}_{\mathbf{AInt}^+}$, the information contained in the subproofs of $\Pi$ is sufficient to obtain a generalized $\textsc{Raint}^+$-subcalculus of $\mathscr{N}\mathscr{D}_{\mathbf{ACl}}$ which constructively completes the information contained in $\mathrm{Theo}([\Pi])$. Let us denote with $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ the abstract calculus $\mathbb{D}(\textsc{Raint}^+, \mathrm{Seq}([\Pi]))$.

| | | | | |
|---|---|---|---|---|
| $\textsc{Id}_1$ : | $\vdash x = x$ | $\in$ | $\textsc{Id}_1(\epsilon)$ | |
| $\textsc{Id}_2$ : | $\Gamma, \Delta \vdash A(t')$ | $\in$ | $\textsc{Id}_2(\Gamma \vdash A(t); \Delta \vdash t = t')$ | |
| $\textsc{Sum}$ : | $\vdash x + 0 = x$ | $\in$ | $\textsc{Sum}(\epsilon)$ | |
| | $\vdash x + Sy = S(x + y)$ | $\in$ | $\textsc{Sum}(\epsilon)$ | |
| $\textsc{Prod}$ : | $\vdash x * 0 = 0$ | $\in$ | $\textsc{Prod}(\epsilon)$ | |
| | $\vdash x * Sy = x * y + x$ | $\in$ | $\textsc{Prod}(\epsilon)$ | |
| $\textsc{Rkp}\vee$ : | $\Gamma, \Delta \vdash \neg A \to B$ | $\in$ | $\textsc{Rkp}\vee_1(\Gamma \vdash B; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$ | with $\neg A \notin \Gamma$ |
| | $\Gamma, \Delta \vdash \neg A \to B$ | $\in$ | $\textsc{Rkp}\vee_1(\Gamma, \neg A \vdash B; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$ | |
| | $\Gamma, \Delta \vdash \neg A \to C$ | $\in$ | $\textsc{Rkp}\vee_2(\Gamma \vdash C; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$ | with $\neg A \notin \Gamma$ |
| | $\Gamma, \Delta \vdash \neg A \to C$ | $\in$ | $\textsc{Rkp}\vee_2(\Gamma, \neg A \vdash C; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$ | |
| | $\neg B \vdash \neg B$ | $\in$ | $\textsc{Rkp}\vee_3(\Delta \vdash (\neg A \to \neg B) \vee (\neg A \to C))$ | |
| | $\neg C \vdash \neg C$ | $\in$ | $\textsc{Rkp}\vee_4(\Delta \vdash (\neg A \to B) \vee (\neg A \to \neg C))$ | |
| $\textsc{Rkp}\exists$ : | $\Gamma, \Delta \vdash \neg A \to B(t)$ | $\in$ | $\textsc{Rkp}\exists_1(\Gamma \vdash B(t); \Delta \vdash \exists x(\neg A \to B(x)))$ | with $\neg A \notin \Gamma$ |
| | $\Gamma, \Delta \vdash \neg A \to B(t)$ | $\in$ | $\textsc{Rkp}\exists_1(\Gamma, \neg A \vdash B(t); \Delta \vdash \exists x(\neg A \to B(x)))$ | |
| | $\neg B(t) \vdash \neg B(t)$ | $\in$ | $\textsc{Rkp}\exists_2(\Delta \vdash \exists x(\neg A \to B(x)))$ | |
| $\textsc{Rcl}$ : | $\Gamma \vdash \forall x A(x)$ | $\in$ | $\textsc{Rcl}(\Gamma \vdash \forall x \neg\neg A(x))$ | |

Table 6: The generalized rule $\textsc{Raint}^+$

The proof of uniform semiconstructivity will be carried out using the following notion of evaluation.

**Definition 6.1** (Neg**-evaluation)** *Let* $\Pi$ *be a set of proofs on* $\mathscr{L}_{\mathbf{PA}}$, *and let* Neg *and* $A$ *be a set of closed negated wff's and a wff in the language* $\mathscr{L}_{\mathbf{PA}}$ *respectively.* $A$ *is* Neg-evaluated in $\Pi$ *iff the following conditions hold:*

- *(i) Either* $A \in \mathrm{Neg}$ *or there exists a proof* $\pi : \Gamma \vdash A \in \Pi$ *with* $\Gamma \subseteq \mathrm{Neg}$*;*
- *(ii) For every closed instance* $\theta A$ *of* $A$*, one of the following conditions holds:*
  - *(a)* $\theta A$ *is atomic or negated;*
  - *(b)* $\theta A \equiv B \wedge C$*, and both* $B$ *and* $C$ *are* Neg*-evaluated in* $\Pi$*;*
  - *(c)* $\theta A \equiv B \vee C$*, and either* $B$ *is* Neg*-evaluated in* $\Pi$ *or* $C$ *is* Neg*-evaluated in* $\Pi$*;*
  - *(d)* $\theta A \equiv B \to C$*, and, for every set* $\mathrm{Neg}'$ *of closed negated wff's of* $\mathscr{L}_{\mathbf{PA}}$ *such that* $\mathrm{Neg}' \supseteq \mathrm{Neg}$*, if* $B$ *is* $\mathrm{Neg}'$*-evaluated in* $\Pi$ *then* $C$ *is* $\mathrm{Neg}'$*-evaluated in* $\Pi$*;*
  - *(e)* $\theta A \equiv \exists x B(x)$*, and* $B(t/x)$ *is* Neg*-evaluated in* $\Pi$ *for some closed term* $t$ *of* $\mathscr{L}_{\mathbf{PA}}$*;*
  - *(f)* $\theta A \equiv \forall x B(x)$*, and, for every closed term* $t$ *of* $\mathscr{L}_{\mathbf{PA}}$*,* $B(t/x)$ *is* Neg*-evaluated in* $\Pi$*.*

The following results, which can be easily proved, will be needed.

**Proposition 6.2** *Let* Neg *be a set of closed negated wff's, let* $\Pi$ *be a set of proofs and let* $A$ *be a wff. If* $A$ *is* Neg*-evaluated in* $\Pi$*, then* $A$ *is* $\text{Neg}'$*-evaluated in* $\Pi$ *for every set of closed negated wff's* $\text{Neg}'$ *including* Neg. □

**Proposition 6.3** *Let* $\Pi$ *be a* CUT*-closed set of proofs, let* Neg *be a set of negated wff's, let* $H$ *be a closed wff and let* $A$ *be an arbitrary wff. If* $A$ *is* $\text{Neg} \cup \{\neg H\}$*-evaluated in* $\Pi$ *and* $\neg H$ *is* Neg*-evaluated in* $\Pi$*, then* $A$ *is* Neg*-evaluated in* $\Pi$. □

**Lemma 6.4** *Let* $\Pi$ *be any set of proofs of* $\mathscr{N}\mathscr{D}_{\mathbf{AInt}^+}$ *and let* Neg *be a set of closed negated wff's of* $\mathscr{L}_{\mathbf{PA}}$*. For any proof* $\pi : \Gamma \vdash H$ *belonging to the closure under substitution of* $[\Pi]$*, if* $\Gamma$ *is* Neg*-evaluated in* $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$*, then* $H$ *is* Neg*-evaluated in* $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$.

*Proof:* Since $\pi$ belongs to the closure under substitution of $[\Pi]$, there exist a proof $\pi' : \Gamma' \vdash H' \in [\Pi]$ and a substitution $\theta$ such that $\theta\Gamma' \vdash \theta H' \equiv \Gamma \vdash H$. Thus, by definition, $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ contains a proof of the sequent $\Gamma' \vdash H'$ and, since it is SUBST-closed, it also contains a proof $\tau' : \Gamma \vdash H$. Now, let $\Gamma = \Delta_0 \cup \{H_1, \ldots, H_n\}$, where $\Delta_0 = \Gamma \cap \text{Neg}$. Since $\Gamma$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ contains proofs $\tau_1 : \Delta_1 \vdash H_1, \ldots, \tau_n : \Delta_n \vdash H_n$ with $\Delta_1 \cup \cdots \cup \Delta_n \subseteq \text{Neg}$. Let $\Delta^* = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_n$; by repeatedly applying the CUT rule to the proofs $\tau', \tau_1, \ldots, \tau_n$, we can construct in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ a proof

$$\tau^* : \Delta^* \vdash H. \tag{6.1}$$

The proof of Point (ii) goes on by induction on depth($\pi$). If depth($\pi$) = 0, the only rule which occurs in $\pi$ is either an assumption introduction or one of the zero-premises rules corresponding to the axioms for identity, sum and product. In the former case the assertion trivially follows. In the latter case, the wff $A$ is atomic; hence, by Point (i) and closure under SUBST of $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, we immediately get the assertion. To prove the induction step, we proceed by cases according to the last rule applied in $\pi$. Here we only treat the representative cases of the rules Ind, wGrz and $\text{KP}_\exists$.

*Induction Rule.*

$$\pi : \Gamma \vdash H \equiv \frac{\pi_1 : \Gamma_1 \vdash B(0) \qquad \pi_2 : \Gamma_2, B(p) \vdash B(S(p))}{\Gamma_1, \Gamma_2 \vdash B(x)}\text{Ind}$$

First of all, the reader can easily prove that, given a closed term $t$ of $\mathscr{L}_{\mathbf{PA}}$, $B(t)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ iff $B(S^n0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, where $S^n0$ is the canonical form of $t$ in **AInt**. Therefore, to prove the assertion it is sufficient to prove that $\theta B(S^h0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ for every closed substitution $\theta$ and every $h \geq 0$. We proceed by induction on $h$; since $\Gamma_1 \subseteq \Gamma$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, $\theta B(0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Now, let us suppose that $\theta B(S^h0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, with $h \geq 0$; then the proof $\theta\pi_2[S^h0/p] : \theta\Gamma_2, \theta B(S^h0/p) \vdash \theta B(S^{h+1}0/p)$ belongs to the closure under substitution of $[\Pi]$ and hence, by the principal induction hypothesis, $B(S^{h+1}0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. This concludes the proof.

*Rule* wGrz.

$$\pi : \Gamma \vdash H \equiv \frac{\pi_1 : \Gamma_1 \vdash \forall x \neg\neg B(x) \qquad \pi_2 : \Gamma_2 \vdash \forall x(B(x) \vee C)}{\Gamma_1, \Gamma_2 \vdash \forall x B(x) \vee C}\text{wGrz}$$

Let $\theta$ be a closed substitution; we must prove that one between the wff's $\theta\forall xB(x)$ and $\theta C$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Let us suppose that $\theta C$ is not Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Since, by induction hypothesis, $\theta\forall x(B(x) \vee C)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, we deduce that, for every closed term $t$ of $\mathscr{L}_{\mathbf{PA}}$, $\theta B(t/x)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. To prove that $\theta\forall xB(x)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, we only need to show that $\theta\forall xB(x)$ is provable from Neg in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. By induction hypothesis on the proof $\pi_1$, a sequent $\Gamma' \vdash \theta\forall x\neg\neg B(x)$, with $\Gamma' \subseteq$ Neg, is provable in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$; since the latter set of proofs is RCL-closed we get that also $\Gamma' \vdash \theta\forall xB(x)$ belongs to $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. This concludes the proof.

*Rule* $\mathrm{KP}_\vee$.

$$\pi : \Gamma \vdash H \equiv \cfrac{\pi_1 : \Gamma, \neg B \vdash C \vee D}{\Gamma \vdash (\neg B \to C) \vee (\neg B \to D)}\,\mathrm{KP}_\vee$$

We must prove that, for every closed substitution $\theta$, one between the wff's $\theta(\neg B \to C)$ and $\theta(\neg B \to D)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Since $\Gamma$ is Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, $\theta\Gamma \cup \{\theta\neg B\}$ is Neg $\cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Then, by the induction hypothesis on the proof $\theta\pi_1$, either $\theta C$ or $\theta D$ is Neg $\cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. For the sake of definiteness, let us assume that $\theta C$ is the evaluated wff. This implies that there exists a proof $\tau : \Delta \vdash \ \theta C \in \mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ with $\Delta \subseteq$ Neg $\cup \{\theta\neg B\}$; we remark that, if $\theta C \in$ Neg $\cup \{\theta\neg B\}$, we can construct $\tau$ as follows

$$\cfrac{\cfrac{\tau^* : \Delta^* \vdash (\neg B \to C) \vee (\neg B \to D)}{\theta\Delta^* \vdash \theta(\neg B \to C) \vee \theta(\neg B \to D)}\,\textsc{Subst}}{\theta C \vdash \theta C}\,\textsc{Rkp}\vee_3$$

where $\tau^*$ is the proof of Point 6.1. Hence, the proof

$$\cfrac{\cfrac{\tau^* : \Delta^* \vdash (\neg B \to C) \vee (\neg B \to D)}{\theta\Delta^* \vdash \theta(\neg B \to C) \vee \theta(\neg B \to D)}\,\textsc{Subst} \qquad \tau : \Delta \vdash \theta C}{\theta\Delta \setminus \{\theta\neg B\}, \Delta^* \vdash \theta(\neg B \to C)}\,\textsc{Rkp}\vee_1$$

belongs to $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. This proves Point (i) of Definition 6.1 for $\theta(\neg B \to C)$. To prove Point (ii) for this wff, let us suppose that $\theta\neg B$ is Neg$'$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, with Neg $\subseteq$ Neg$'$. We already know that $\theta C$ is Neg $\cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, and hence, by Proposition 6.2, it is also Neg$' \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Since $\theta\neg B$ is Neg$'$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$, by Proposition 6.3 we have that $\theta C$ is Neg$'$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. This concludes the proof.

□

**Corollary 6.5** *Let* $\Pi$ *be any set of proofs of* $\mathscr{N}\mathscr{D}_{\mathbf{AInt}^+}$. *The set* Theo($[\Pi]$) *is semiconstructive in* Theo($\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$).

*Proof:* If $A \vee B$ is a closed wff in Theo($[\Pi]$), then there exists a proof $\pi : \vdash A \vee B$ in the closure under substitution of $[\Pi]$. Since the empty set of premises is trivially Neg-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$ with Neg $= \emptyset$ by Lemma 6.4, $A \vee B$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. Thus, at least one of the wff's $A$ and $B$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$; by Definition 6.1, this means that one between the sequents $\vdash A$ and $\vdash B$ is provable in $\mathbb{D}_{\mathbf{AInt}^+}([\Pi])$. The proof of the (wED) property is similar. □

Since RAINT$^+$ is a non-increasing generalized rule, $\mathscr{ND}_{\mathbf{ACl}}$ is uniformly RAINT$^+$-closed and the previous corollary holds, we get:

**Theorem 6.6** *$\mathscr{ND}_{\mathbf{AInt}^+}$ is a uniformly semiconstructive calculus in $\mathscr{ND}_{\mathbf{ACl}}$.* □

Now, let us denote with $\mathbf{AInt}^+$ the intermediate **PA**-system included in $\mathbf{Constr}_2(\mathbf{PA})$ coinciding with Theo($\mathscr{ND}_{\mathbf{AInt}^+}$). To conclude the presentation of this example, we will prove that $\mathbf{AInt}^+$ cannot be extended into a recursively enumerable and constructive **T**-system with **T** any theory including **PA**.

**Theorem 6.7** *There exists no consistent and recursively axiomatizable constructive* **T**-*system* **S** *such that* $\mathbf{PA} \subseteq \mathbf{T}$ *and* $\mathbf{AInt}^+ \subseteq \mathbf{S}$.

*Proof:* Let **S** be a constructive recursively axiomatizable **T**-system including $\mathbf{AInt}^+$ (with $\mathbf{PA} \subseteq \mathbf{T}$). Let $p(x)$ be a unary recursively enumerable but not recursive predicate. By the Normal Form Theorem, there exists a primitive recursive binary predicate $q(x, y)$ such that $p(x) \leftrightarrow \exists y q(x, y)$. Now, by well known representability results, there exists a wff $A(x, y)$ with two free variables of $\mathscr{L}_{\mathbf{PA}}$ which strongly represents the predicate $q(x, y)$ in **AInt**; this means that $\forall x \forall y(A(x, y) \vee \neg A(x, y)) \in \mathbf{AInt}$, and, for every $a, b \in \mathbf{N}$, denoting with $\widetilde{a}$ and $\widetilde{b}$ the corresponding numerals, if $q(a, b)$ is true then $A(\widetilde{a}, \widetilde{b}) \in \mathbf{AInt}$, and $\neg A(\widetilde{a}, \widetilde{b}) \in \mathbf{AInt}$ if $q(a, b)$ is false. Moreover, let $G$ be a closed wff of $\mathscr{L}_{\mathbf{PA}}$ such that $G \notin \mathbf{S}$ and $\neg G \notin \mathbf{S}$ (such a wff exists by the intuitionistic version of Gödel Incompleteness Theorem, since **S** is recursively axiomatizable, see, e.g., [Troelstra, 1973]). Now, let us consider the wff $H(x) \equiv \exists y A(x, y) \vee \forall y(\neg A(x, y) \vee (G \vee \neg G))$. It is easy to check that $H(x)$ is provable in $\mathbf{AInt}^+$. We will show that one can effectively decide whether $p(k)$ holds or not, for every natural number $k$. Indeed, since **S** is constructive and recursively axiomatizable, there is a terminating effective procedure which, for every input $k \in \mathbf{N}$, outputs either a **S**-proof of $\exists y A(\widetilde{k}, y)$ or a **S**-proof of $\forall y(\neg A(\widetilde{k}, y) \vee (G \vee \neg G))$. Now, if $\exists y A(\widetilde{k}, y) \in \mathbf{S}$, then, by the constructivity of **S**, $A(\widetilde{k}, \widetilde{b}) \in \mathbf{S}$ for some numeral $\widetilde{b}$; this implies that $\exists y q(k, y)$ holds. On the other hand, if $\forall y(\neg A(\widetilde{k}, y) \vee (G \vee \neg G)) \in \mathbf{S}$, then, since, by the hypotheses on the wff $G$, $G \vee \neg G \notin \mathbf{S}$, we deduce that $\neg A(\widetilde{k}, \widetilde{b}) \in \mathbf{S}$, for every $b \in \mathbf{N}$; this implies that $\exists y q(k, y)$ does not hold. Since $p(x) \leftrightarrow \exists y q(x, y)$, we have that $p(x)$ is recursive, against the assumptions. □

## 6.2 A uniformly semiconstructive PA-system included in $\mathbf{Constr}_1(\mathbf{PA})$

Now, let us consider the **PA**-system $\mathbf{AInt}^{++}$ obtained by adding to Intuitionistic Arithmetic **AInt** the axiom schema (DT) of Table 2.

$\mathscr{ND}_{\mathbf{AInt}^{++}}$ will denote the calculus for $\mathbf{AInt}^{++}$ obtained by adding the zero premises rule

$$\frac{}{\vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)}\ \text{DT}$$

to the calculus $\mathscr{ND}_{\mathbf{AInt}}$.

Now, we denote with RAINT$^{++}$ the union of the generalized rules CUT and SUBST, of the generalized rules ID, SUM, PROD of Table 6, and of the rules of Table 7. It is easy to check that RAINT$^{++}$ is non-increasing and that the calculus $\mathscr{ND}_{\mathbf{ACl}}$ is uniformly RAINT$^{++}$-closed.

| | | | |
|---|---|---|---|
| $\textsc{Rdt}_1$ : | $\vdash \exists x A(x)$ | $\in$ | $\textsc{Rdt}_1(\vdash A(t); \vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B))$ |
| $\textsc{Rdt}_2$ : | $\vdash \forall x(A(x) \to B \vee \neg B)$ | $\in$ | $\textsc{Rdt}_2(\vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B))$ |

Table 7: The generalized rule $\textsc{Raint}^{++}$

Let us denote with $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$ the abstract calculus $\mathbb{D}(\textsc{Raint}^{++}, \mathrm{Seq}([\Pi]))$. The proof of uniform semiconstructivity of $\mathcal{ND}_{\mathbf{AInt}^{++}}$ can be carried out along the lines of the proof given in the previous section, but using the simpler notion of evaluation of Definition 5.1. Hence the main lemma is:

**Lemma 6.8** *Let $\Pi$ be any set of proofs of $\mathcal{ND}_{\mathbf{AInt}^{++}}$. For any proof $\pi : \Gamma \vdash H$ belonging to the closure under substitution of $[\Pi]$, if $\Gamma$ is evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$, then $H$ is evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$.*

In the proof we must consider the case where $\mathrm{depth}(\pi) = 0$ and the only rule applied in $\pi$ is DT. In this case $\Gamma$ is empty and $H \equiv \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)$. Let us consider a closed instance $\theta H$ of this wff, and let us suppose that $\theta \exists x A(x)$ is not evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$. Then, for every closed term $t$ of $\mathscr{L}_{\mathbf{PA}}$, $\theta A(t)$ is not evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$ (we remark that, by the $\textsc{Rdt}_1$-closure of $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$, the case where some $\theta A(t)$ is evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$ and $\theta \exists x A(x)$ is not provable in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$ cannot arise). But, since, for every closed term $t$ of $\mathscr{L}_{\mathbf{PA}}$, $\theta A(t)$ is not evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$ and the latter set of proofs is $\textsc{Rdt}_2$-closed, we immediately deduce that $\theta \forall x(A(x) \to B \vee \neg B)$ is evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$. Hence $H$ is evaluated in $\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi])$.

From the previous lemma, we get:

**Corollary 6.9** *Let $\Pi$ be any set of proofs of $\mathcal{ND}_{\mathbf{AInt}^{++}}$. Then the set* Theo$([\Pi])$ *is semiconstructive in* Theo$(\mathbb{D}_{\mathbf{AInt}^{++}}([\Pi]))$. □

Since $\textsc{Raint}^{++}$ is a non-increasing generalized rule, $\mathcal{ND}_{\mathbf{ACl}}$ is uniformly $\textsc{Raint}^{++}$-closed and the previous corollary holds, we get:

**Theorem 6.10** *$\mathcal{ND}_{\mathbf{AInt}^{++}}$ is a uniformly semiconstructive calculus in $\mathcal{ND}_{\mathbf{ACl}}$.* □

We point out that the principle (St$\exists$) of Table 2 (holding in $\mathbf{Constr}_1(\mathbf{PA})$) is derivable from (DT) and (Kur). On the other hand we can add to $\mathbf{AInt}^{++}$ the rules Mk and Kur of §5 without affecting its uniform semiconstructivity (and without extending the generalized rule $\textsc{Raint}^{++}$). However, we can prove that $\mathbf{AInt}^{++}$ cannot be extended into a recursively enumerable and constructive $\mathbf{T}$-system with $\mathbf{PA} \subseteq \mathbf{T}$.

**Theorem 6.11** *There exists no consistent and recursively axiomatizable constructive* **T**-*system* **S** *such that* $\mathbf{PA} \subseteq \mathbf{T}$ *and* $\mathbf{AInt}^{++} \subseteq \mathbf{S}$.

*Proof:* Let **S** be a recursively axiomatizable and constructive **T**-system including $\mathbf{AInt}^{++}$ (with $\mathbf{PA} \subseteq \mathbf{T}$). We will show that, for every closed wff $A$, one can decide whether $A \in \mathbf{S}$ or not. Indeed, let $G$ be a closed wff of $\mathscr{L}_{\mathbf{PA}}$ such that $G \notin \mathbf{S}$ and $\neg G \notin \mathbf{S}$ (such a wff exists by the intuitionistic version of Gödel Incompleteness Theorem [Troelstra, 1973]). Since **S** is constructive and recursively axiomatizable and, for every closed wff $A$, $A \vee (A \to G \vee \neg G) \in \mathbf{AInt}^{++}$, there is a terminating effective procedure which, taking any closed wff $A$ of $\mathscr{L}_{\mathbf{PA}}$ as an input, outputs either a **S**-proof of $A$ or a **S**-proof of $A \to G \vee \neg G$. Now, if $A \to G \vee \neg G \in \mathbf{S}$, by the choice of $G$ and the constructivity of **S**, $A \notin \mathbf{S}$. Hence, the set of theorems of **S** is recursive, against the Intuitionistic version of Church's Theorem. $\square$

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