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Extracting information from intermediate semiconstructive HA-systems

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Abstract

In this paper we will study the problem of uniformly extracting information from proofs in semiconstructive calculi, a kind of calculi which is of interest in the framework of program synthesis. Here we will discuss the notion of uniformly constructive calculus, we introduce our information extraction mechanism and we apply it to two calculi extending Intuitionistic Arithmetic.

Keywords: intermediate semi-constructive systems, information extraction

Contents

1 Introduction

In previous works [Ferrari, 1997; Ferrari et al., 1999a; Ferrari et al., 1999b] the authors have developed a method to extract information from proofs of constructive systems that works also in cases where the usual information extraction techniques based on Normalization, Cut-elimination or Realizability cannot be applied.

In this paper we extend our technique to handle "weaker" systems, we call *semiconstructive*. Formally, let us consider a system $\mathbf{T} \oplus \mathbf{L}$, where $\mathbf{T}$ is a first order theory (the mathematical part) and $\mathbf{L}$ is a superintuitionistic logic (the deductive apparatus). $\mathbf{T} \oplus \mathbf{L}$ is *semiconstructive* if it satisfies the *weak disjunction property* (if a closed wff $A \vee B$ belongs to $\mathbf{T} \oplus \mathbf{L}$ then either A or B belongs to the corresponding classical theory $\mathbf{T} \oplus \mathbf{Cl}$) and the *weak explicit definability property* (if a closed wff $\exists x A(x)$ belongs to $\mathbf{T} \oplus \mathbf{L}$ then $A(t)$ belongs to the corresponding classical theory $\mathbf{T} \oplus \mathbf{Cl}$ for some closed term t).

Now, the notion of semiconstructive system is relevant in the context of the authors' approach to program synthesis and Abstract Data Types specification [Miglioli and Ornaghi, 1981; Miglioli et al., 1989; Miglioli et al., 1994; Avellone et al., 1999]. In fact, if $\mathbf{T}$ is a theory completely formalizing an Abstract Data Type, according to the characterization of Abstract Data Types based on the notion of *isoinitial model* [Miglioli et al., 1994], the addition of $\mathbf{T}$ to a semiconstructive deductive apparatus $\mathbf{L}$ gives rise to a recursively axiomatizable and semiconstructive system $\mathbf{T} \oplus \mathbf{L}$. Therefore, if $\mathbf{T} \oplus \mathbf{L}$ contains a proof π of a formula $\forall \underline{x} \exists ! y A(\underline{x}, y)$ (respectively, a formula of the kind $\forall \underline{x}(B(\underline{x}) \vee \neg B(\underline{x}))$), then the whole system $\mathbf{T} \oplus \mathbf{Cl}$ can be used to compute the function (respectively, the predicate) associated with such a formula [Miglioli et al., 1989; Miglioli et al., 1994]. But, if the system $\mathbf{T} \oplus \mathbf{L}$ does not satisfy further properties, the algorithm to compute the function (the predicate) is highly inefficient since it is based on an enumeration of the theorems of $\mathbf{T} \oplus \mathbf{Cl}$; moreover such an algorithm does not use the "local" information contained in the proof π (the proof π is only used to guarantee its termination). For this reason we introduce the notion of *uniformly semiconstructive system*. If $\mathbf{T} \oplus \mathbf{L}$ is such a system, the formula (the predicate) related to the proof π can be computed by searching a calculus, the *extraction calculus for* π, whose proofs are generate starting from the formulas contained in π by means of *extraction rules*; moreover, the proof of the extraction calculus have a bounded logical complexity depending on the proof π in hand.

According to this property, one of our goals is to provide tools to significantly extend the field of applications of the traditional proof-theoretic techniques, yet providing a good paradigm of proofs-as-programs in the context of semiconstructive calculi.

In this paper, after having introduced the information extraction mechanism and the notion of uniformly constructive system, we will study two systems extending Intuitionistic Arithmetic that turn out to be uniformly semiconstructive and that cannot be extended in recursively enumerable constructive systems.

2 Preliminaries

Let Σ be an extra-logical alphabet; the the set of *terms* and the set of *well formed formulas* (*wff's* for short) of the language $\mathscr{L}_\Sigma$ are built up in the usual way, starting from Σ, a denumerable set $\mathscr{V}$ of individual variables and the logical constants $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$ (we consider $\neg A$ as an abbreviation for $A \rightarrow \bot$). The notions of *degree* $\mathrm{dg}(A)$ of a wff A, *substitution* θ and the notion of *sequent* are the usual ones.

The natural calculus $\mathscr{ND}_{\mathbf{Int}}$ for first-order intuitionistic logic with identity is given in Table 1, while the calculus for first-order Classical $\mathscr{ND}_{\mathbf{Cl}}$ is obtained by replacing the rule $\bot_{\mathrm{Int}}$ of the calculus $\mathscr{ND}_{\mathbf{Int}}$ with the rule

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{Cl}}$$

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\,\mathrm{W} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\bot_{\mathrm{Int}} \quad \text{where } A \text{ is an atomic wff.}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\mathrm{I}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\mathrm{E}\wedge \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\mathrm{E}\wedge$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\mathrm{I}\vee \qquad \frac{\Gamma \vdash A \vee B \quad \Delta, A \vdash C \quad \Theta, B \vdash C}{\Gamma, \Delta, \Theta \vdash C}\,\mathrm{E}\vee$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\mathrm{I}\to \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}\,\mathrm{E}\to$$

$$\frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A(x)}\,\mathrm{I}\forall \quad \text{where } y \text{ does not occur free in } \Gamma \text{ or } \forall x A(x). \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t/x)}\,\mathrm{E}\forall$$

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A(x)}\,\mathrm{I}\exists \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Delta, A(y/x) \vdash C}{\Gamma, \Delta \vdash C}\,\mathrm{E}\exists \quad \text{where } y \text{ does not occur free in } \Delta,\ \exists x A(x) \text{ or } C.$$

$$\frac{}{\vdash x = x}\,\mathrm{id}_1 \qquad \frac{\Gamma \vdash A(t/x) \quad \Delta \vdash t = t'}{\Gamma, \Delta \vdash A(t'/x)}\,\mathrm{id}_2 \quad \text{where } A(x) \text{ is an atomic wff.}$$

Table 1: The calculus $\mathscr{ND}_{\mathbf{HA}}$

Given the the usual extra-logical alphabet of arithmetic $\mathscr{A} = \{0, S, +, *\}$, the natural calculi $\mathscr{ND}_{\mathbf{HA}}$ and $\mathscr{ND}_{\mathbf{PA}}$ for Intuitionistic and Classical Arithmetic can be obtained by adding the rules of Table 2 to $\mathscr{ND}_{\mathbf{Int}}$ and $\mathscr{ND}_{\mathbf{Cl}}$ respectively.

We assume the usual convention on *proper parameters* and *free variables* of the natural deduction rules stated in [Prawitz, 1965; Troelstra, 1973], in such a way to guarantee that the tree-structure $\theta\pi$ obtained by replacing some of the free variables of a proof π with terms is a well defined proof.

Now, since we aim to study systems obtained by enlarging the intuitionistic system with logical and mathematical principles, we need to introduce a formal definition of this kind of systems. We denote with **Int** (**Cl**) the set of intuitionistically (classically) valid wff's of the pure first-order language $\mathscr{L}$. A *(first-order) intermediate pseudo-logic* is any set of wff's **L** such that: $\mathbf{Int} \subseteq \mathbf{L} \subseteq \mathbf{Cl}$ and **L** is closed under modus ponens and generalization. On the other hand, an *intermediate logic* **L** is an intermediate pseudo logic closed under predicate substitution (see, e.g., [Ono, 1972]).

Passing from the pure first-order language $\mathscr{L}$ to $\mathscr{L}_\Sigma$ (where the relation declarations of Σ are seen as *constant relation declarations*, and hence predicate substitutions are not allowed), one can easily define $\mathbf{Int}_\Sigma$ ($\mathbf{Cl}_\Sigma$) as the subset of $\mathscr{L}_\Sigma$ obtained by correctly

$$\frac{\Gamma \vdash 0 = S(x)}{\Gamma \vdash \bot}\,\mathrm{S}_1 \qquad \frac{\Gamma \vdash S(t) = S(t')}{\Gamma \vdash t = t'}\,\mathrm{S}_2$$

$$\frac{}{\Gamma \vdash t + 0 = t}\,+_1 \qquad \frac{}{\Gamma \vdash t + S(t') = S(t + t')}\,+_2$$

$$\frac{}{\Gamma \vdash t * 0 = 0}\,*_1 \qquad \frac{}{\Gamma \vdash t * S(t') = t * t' + t}\,*_2$$

$$\frac{\Gamma \vdash B(0) \quad \Delta, B(y) \vdash B(S(y))}{\Gamma, \Delta \vdash B(x)}\,\mathrm{Ind} \quad \text{where } y \text{ does not occur free in } \Delta.$$

Table 2: Rules for Arithmetic

substituting the predicate variables with wff's of $\mathscr{L}_\Sigma$ in the wff's of **Int** (**Cl**). A pseudo-logic $\mathbf{L}_\Sigma$ will be any subset of $\mathscr{L}_\Sigma$ such that $\mathbf{Int}_\Sigma \subseteq \mathbf{L}_\Sigma \subseteq \mathbf{Cl}_\Sigma$, and $\mathbf{L}_\Sigma$ is closed under modus ponens and generalization; when Σ is understood, we omit to indicate the subscript Σ. Finally, if Γ is a set of classically valid wff's of $\mathscr{L}_\Sigma$, we denote with $\Gamma \oplus \mathbf{L}$ the smallest set of wff's (which is an intermediate pseudo-logic) closed under modus ponens and generalization containing the intermediate pseudo-logic $\mathbf{L}$ and Γ.

In this framework, given a Σ-theory $\mathbf{T}$ (i.e. a recursively enumerable set of classically consistent wff's of $\mathscr{L}_\Sigma$), we call *(intermediate)* $\mathbf{T}$*-system* any set $\mathbf{S} \subseteq \mathscr{L}_\Sigma$ such that $\mathbf{T} \oplus \mathbf{Int} \subseteq \mathbf{S} \subseteq \mathbf{T} \oplus \mathbf{Cl}$ and $\mathbf{S}$ is closed under modus ponens and generalization.

Given $\Gamma, \Delta \subseteq \mathscr{L}_\Sigma$ such that $\Gamma \subseteq \Delta$, we say that Γ is *semiconstructive in* Δ iff the *weak disjunction property* (wDp) and the *weak explicit definability property* (wEd) hold:

(wDp): if $A \vee B \in \Gamma$ and $A \vee B$ is a closed wff, then either $A \in \Delta$ or $B \in \Delta$.

(wEd): if $\exists x A(x) \in \Gamma$ and $\exists x A(x)$ is a closed wff, then $A(t/x) \in \Delta$ for some closed term t of the language.

We simply say that a $\mathbf{T}$-system $\mathbf{S}$ is *semiconstructive* if $\mathbf{S}$ is semiconstructive in $\mathbf{T} \oplus \mathbf{Cl}$.

The *full* disjunction property (Dp) and the *full* explicit definability property (Ed) characterizing our notion of *constructive* $\mathbf{T}$*-system* can be obtained by imposing $\Gamma = \Delta$ in (wDp) and (wEd).

3 The information extraction mechanism

In this section we will provide a short presentation of our mechanism to extract information from proofs, giving only the main definitions and results (for a complete discussion see [Ferrari, 1997; Ferrari et al., 1999b]). We remark that, even if in this paper all the systems are presented by means of pseudo-natural deduction systems, the extraction mechanism is based on an abstract definition of a calculus allowing to treat also extraction from Gentzen-style, Tableau-style or Hilbert-style calculi.

First of all we define a *proof* on a language $\mathscr{L}_\Sigma$ as any finite object π such that:

(π.1) The (finite) set of wff's of $\mathscr{L}_\Sigma$ occurring in π is uniquely determined and nonempty;

(π.2) The sequent $\Gamma \vdash \Delta$ proved by π is uniquely determined, where Γ and Δ are finite sets of wff's of $\mathscr{L}_\Sigma$. Γ (possibly empty) is the set of *assumptions* of π while Δ, which must be nonempty, is the set of *consequences* of π.

Proofs are characterized by the following attributes: Seq(π) indicates the sequent $\Gamma \vdash \Delta$ proved by π, Wffs(π) denotes the set of wff's of $\mathscr{L}_\Sigma$ occurring in π, and dg(π) denotes the *degree of* π which is the maximum among the degrees of the wff's occurring in π. The compact notation $\pi : \Gamma \vdash \Delta$ will be used to indicate that $\mathrm{Seq}(\pi) = \Gamma \vdash \Delta$. Moreover, the degree of a sequent $\Gamma \vdash A$, denoted by $\mathrm{dg}(\Gamma \vdash \Delta)$, is the maximum among the degrees of the wff's occurring in $\Gamma \cup \Delta$.

A *calculus* on $\mathscr{L}_\Sigma$ is a pair $\mathbf{C} = (C, [\cdot])$, where C is a recursive set of proofs on the language $\mathscr{L}_\Sigma$ and $[\cdot]$ is a recursive map from C to the set of finite subsets of C with the following properties:

(**C**.1) $\pi \in [\pi]$;

(**C**.2) For every $\pi' \in [\pi]$, $[\pi'] \subseteq [\pi]$;

(**C**.3) For every $\pi' \in [\pi]$, $\mathrm{dg}(\pi') \leq \mathrm{dg}(\pi)$.

The map $[\cdot]$ associates with every proof of the calculus the set of its *relevant* subproofs; conditions (**C**.2) and (**C**.3) constitute a natural formalization of the notion of subproof. We remark that any usual inference system (Hilbert-style, Gentzen-style, ...) is a calculus according to our definition. In particular $\mathscr{N}\mathscr{D}_{\mathbf{Int}}$ is a calculus in this sense, where we consider $[\pi]$ to be the usual set of subproofs of π.

To simplify the notation we will identify a calculus $\mathbf{C}$ with the set of its proofs. Now, given $\Pi \subseteq \mathbf{C}$, $[\Pi]$ denotes the *closure under subproofs* of Π in $\mathbf{C}$; namely $[\Pi] = \{\pi' :$ there exists $\pi \in \Pi$ such that $\pi' \in [\pi]\}$.

We associate with each $\Pi \subseteq \mathbf{C}$ the following attributes: $\mathrm{Seq}(\Pi) = \cup_{\pi\in\Pi}\mathrm{Seq}(\pi)$; dg($\Pi$) is the *degree* of Π, i.e. $\mathrm{dg}(\Pi) = \max\{\mathrm{dg}(\pi) : \pi \in \Pi\}$, where $\mathrm{dg}(\Pi) = \infty$ if Π contains proofs of any complexity; $\mathrm{Theo}(\Pi) = \{A : \vdash A \in \mathrm{Seq}(\Pi)\}$ is the set of *theorems proved in* Π.

In the following we will be interested in characterizing subsets of a calculus which have some closure properties, to this aim we introduce the notion of generalized rule. Given a language $\mathscr{L}_\Sigma$, let Ξ be the set of all the sequents on $\mathscr{L}_\Sigma$ and let Ξ^* be the set of all the finite sequences (with ϵ denoting the empty sequence) of sequents in Ξ; a *generalized rule (on $\mathscr{L}_\Sigma$)* is a relation $\mathscr{R} \subseteq \Xi^* \times \Xi$. We will write $\sigma \in \mathscr{R}(\sigma^*)$ as a shorthand for $(\sigma^*, \sigma) \in \mathscr{R}$.

A set of sequents S is *$\mathscr{R}$-closed* iff, for every $\sigma, \sigma_1, \ldots, \sigma_n \in \Xi$, if $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$ and $\sigma_1, \ldots, \sigma_n \in S$ then $\sigma \in S$; a set of proofs is $\mathscr{R}$-closed iff Seq(Π) is $\mathscr{R}$-closed.

Examples of generalized rules we will use in the following are:

- *Substitution rule* SUBST: its domain is the set of all the sequents, and, for every substitution θ, $\theta\Gamma \vdash \theta\Delta \in \mathrm{SUBST}(\Gamma \vdash \Delta)$.

- *Intuitionistic Cut rule* CUT: its domain contains all the sequences of sequents of the kind $\Gamma_1 \vdash H; \Gamma_2, H \vdash A$, and $\Gamma_1, \Gamma_2 \vdash A \in \mathrm{CUT}(\Gamma_1 \vdash H; \Gamma_2, H \vdash A)$.

It is immediate to check that every pseudo-natural deduction calculus including $\mathscr{N}\mathscr{D}_{\mathbf{Int}}$ is CUT-closed and SUBST-closed.

A generalized rule $\mathscr{R}$ is an *extraction rule for* $\mathbf{C}$ (*e-rule* for short) with respect to a positive integer h and a function $\phi : \mathbf{N} \to \mathbf{N}$ if:

($\mathscr{R}$.1) $\mathbf{C}$ is $\mathscr{R}$-closed;

($\mathscr{R}$.2) For every $\sigma \in \mathscr{R}(\epsilon)$, $\mathrm{dg}(\sigma) \leq h$;

($\mathscr{R}$.3) For every $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$ such that $\pi_1 : \sigma_1, \ldots, \pi_n : \sigma_n$ belong to $\mathbf{C}$, there exists a proof $\pi : \sigma \in \mathbf{C}$ such that:

$$\mathrm{dg}(\pi) \leq \max\{\mathrm{dg}(\pi_1), \ldots, \mathrm{dg}(\pi_n), \phi(\mathrm{dg}(\sigma_1)), \ldots, \phi(\mathrm{dg}(\sigma_n)), \phi(\mathrm{dg}(\sigma))\}.$$

We remark that condition ($\mathscr{R}$.1) says that $\mathscr{R}$ must be an admissible rule for $\mathbf{C}$, while conditions ($\mathscr{R}$.2) and ($\mathscr{R}$.3) require that $\mathscr{R}$ must be simulated in a uniform way (w.r.t. the degrees) in the calculus $\mathbf{C}$. It is easy to check that Cut and Subst are e-rules for $\mathscr{N}\mathscr{D}_{\mathbf{Int}}$.

Now, given a recursive generalized rule $\mathscr{R}$ (on $\mathscr{L}_\Sigma$) and a recursive set Π of proofs of $\mathbf{C}$ (on $\mathscr{L}_\Sigma$), the *extraction calculus for* Π is the calculus $\mathbb{D}(\mathscr{R}, [\Pi])$ defined as follows:

(i) If $\sigma \in \mathrm{Seq}([\Pi])$, then $\tau \equiv \sigma$ is a proof-tree of $\mathbb{D}(\mathscr{R}, [\Pi])$ with root σ and $\mathrm{depth}(\tau) = 1$.

(ii) If $\tau_1 : \sigma_1, \ldots, \tau_n : \sigma_n$ are proof-trees of $\mathbb{D}(\mathscr{R}, [\Pi])$ (where σ_i is the root of τ_i) then, for every $\sigma \in \mathscr{R}(\sigma_1; \ldots; \sigma_n)$, the proof-tree

$$\tau \equiv \frac{\tau_1 : \sigma_1 \ \ldots \ \tau_n : \sigma_n}{\sigma}\mathscr{R}$$

with root σ belongs to $\mathbb{D}(\mathscr{R}, [\Pi])$ and $\mathrm{depth}(\tau) = \max\{\mathrm{depth}(\tau_1), \ldots, \mathrm{depth}(\tau_n)\}+1$.

Given two calculi $\mathbf{C}_1$ and $\mathbf{C}_2$, we say that $\mathbf{C}_1$ *is uniformly embedded in* $\mathbf{C}_2$ if there exist a mapping $r : \mathbf{C}_1 \to \mathbf{C}_2$ and a function $\psi : \mathbf{N} \to \mathbf{N}$ such that, for every $\pi : \sigma \in \mathbf{C}_1$, $r(\pi) : \sigma \in \mathbf{C}_2$ and $\mathrm{dg}(r(\pi)) \leq \psi(\mathrm{dg}(\pi))$. In [Ferrari, 1997; Ferrari et al., 1999b] the following properties of the extraction calculi are proved:

Theorem 3.1 *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be two calculi over the same language* $\mathscr{L}_\Sigma$ *such that* $\mathbf{C}_1$ *is uniformly embedded in* $\mathbf{C}_2$. *Let* $\mathscr{R}$ *be an e-rule for* $\mathbf{C}_2$ *(w.r.t.* h *and* ϕ*) and let* Π *be a recursive subset of* $\mathbf{C}_1$ *with* $\mathrm{dg}(\Pi) \leq k$ *(*$k \geq 0$*). Then:*

1. *For every proof* τ *in* $\mathbb{D}(\mathscr{R}, [\Pi])$, $\mathrm{dg}(\tau) \leq \max\{h, k\}$*;*

2. $\mathbb{D}(\mathscr{R}, [\Pi]))$ *is uniformly embedded in* $\mathbf{C}_2$.

Now, we give the fundamental definition of uniformly semiconstructive calculus:

Definition 3.2 *Let* $\mathbf{C}_1 = (C_1, [.]_1)$ *and* $\mathbf{C}_2 = (C_2, [.]_2)$ *be two calculi on the same language* $\mathscr{L}_\Sigma$ *such that* $\mathbf{C}_1$ *is uniformly embedded in* $\mathbf{C}_2$. $\mathbf{C}_1$ *is* uniformly semiconstructive *in* $\mathbf{C}_2$ *iff there exists an e-rule* $\mathscr{R}$ *for* $\mathbf{C}_2$ *such that, for every recursive subset* Π *of* $\mathbf{C}_1$, $\mathrm{Theo}([\Pi]_1)$ *is semiconstructive in* $\mathrm{Theo}(\mathbb{D}(\mathscr{R}, [\Pi]_1))$.

The main effect of the previous definition comes from Point (1) of Theorem 3.1. Indeed, this assures that, if $\pi : \vdash \exists x A(x) \in \mathbf{C}_1$, then we can "semiconstructively complete" the information contained in the proof π by means of the calculus $\mathbb{D}(\mathscr{R}, [\pi]_1)$ (e.g., $\mathbb{D}(\mathscr{R}, [\pi]_1)$ proves a sequent of the kind $\vdash A(t)$ for some term t) and $\mathbb{D}(\mathscr{R}, [\pi]_1)$ has bounded logical complexity. Moreover, by Point (2) of Theorem 3.1, $\mathbf{C}_2$ itself contains a logically bounded

($\mathscr{R}$-closed) set of proofs which "semiconstructively completes" the information contained in π.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two calculi generating respectively a **T**-system **S** (i.e., $\text{Theo}(\mathbf{C}_1) = \mathbf{S}$) and its classical extension $\mathbf{T}\oplus\mathbf{Cl}$ (i.e., $\text{Theo}(\mathbf{C}_2) = \mathbf{T}\oplus\mathbf{Cl}$). We say that **S** is *uniformly semiconstructive* if $\mathbf{C}_1$ is uniformly semiconstructive in $\mathbf{C}_2$.

To complete this section we also give the stronger definition of uniformly constructive calculus.

Definition 3.3 *A calculus* $\mathbf{C} = (C, [.])$ *is* uniformly constructive *if there exists an e-rule* $\mathscr{R}$ *for* $\mathbf{C}$ *such that, for every recursive* $\Pi \subseteq \mathbf{C}$*,* $\text{Theo}(\mathbb{D}(\mathscr{R}, [\Pi]))$ *is constructive.*

Obviously a **T**-system **S** is uniformly constructive if it can be generated by a uniformly constructive calculus **C**.

Using the latter characterization the authors have shown in [Ferrari et al., 1999b] that a wide family of systems $\mathbf{S} = \mathbf{T} + \mathbf{L}$ (where **T** is a mathematical theory and **L** is a superintuitionistic calculus) are uniformly constructive. Namely, in [Ferrari et al., 1999b] the authors have shown that several systems **S** involving an Harrop theory **T** and superintuitionistic (intermediate) logics **L** are uniformly constructive. The most representative principles studied in that paper are: the *Grzegorczyk Principle* $\forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B$ with $x \notin \text{FV}(B)$, the *Kuroda Principle* $\forall x \neg\neg A(x) \rightarrow \neg\neg\forall x A(x)$, the *Extended Scott Principle* $(\forall x(\neg\neg A(x) \rightarrow A(x)) \rightarrow \exists x(A(x) \vee \neg A(x))) \rightarrow \exists x(\neg A(x) \vee \neg\neg A(x))$, the *Kreisel-Putnam Principle* $(\neg A \rightarrow B \vee C) \rightarrow (\neg A \rightarrow B) \vee (\neg A \rightarrow C)$ and the *Independence of Premises Principle* $(\neg A \rightarrow \exists x B(x)) \rightarrow \exists x(\neg A \rightarrow B(x))$ with $x \notin \text{FV}(A)$.

On the other hand, in [Ferrari et al., 1999a] the authors have considered systems **S** involving *Hereditary Harrop Theories*, *Grzegorczyk Principle* and the *Descending Chain Principle* $\exists x A(x) \wedge \forall y(A(y) \rightarrow \exists z((A(z) \wedge z < y) \vee B)) \rightarrow B$, showing that in such cases goal-oriented e-rules can be applied to define the abstract calculus.

4 Uniformly semiconstructive HA-systems

In this section we will investigate the uniform semiconstructivity of two calculi including Intuitionistic Arithmetic (but we might as well consider a more general family of calculi involving arbitrary Harrop theories, Cover Set Inductions and Descending Chain Principles). The systems studied in this section are particularly interesting since they contain principles which, as far as we know, cannot be treated by the usual information extraction techniques based on Normalization, Cut elimination or Realizability.

4.1 The uniformly semiconstructive HA-system HA$^+$

In this section we will discuss the uniformly semiconstructive system $\mathbf{HA}^+$ obtained by adding to Intuitionistic Arithmetic **HA** the following principles:

$$
\begin{array}{ll}
(\text{Kur}) & \forall x \neg\neg A(x) \rightarrow \neg\neg\forall x A(x) \\
(\text{KP}_\vee) & (\neg A \rightarrow B \vee C) \rightarrow (\neg A \rightarrow B) \vee (\neg A \rightarrow C) \\
(\text{KP}_\exists) & (\neg A \rightarrow \exists x B(x)) \rightarrow \exists x(\neg A \rightarrow B(x)) \\
(\text{wGrz}) & \forall x \neg\neg A(x) \wedge \forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B \quad \text{with } x \notin \text{FV}(B)
\end{array}
$$

The principle (Kur), known as *Kuroda Principle*, plays an important role with respect to Classical Logic. Indeed, a theory **T** is classically consistent iff it is consistent in any intermediate predicate logic including Kuroda Principle [Gabbay, 1981]. As a consequence of this feature, and as it is well known, the kurodian (and super-kurodian) formal systems are not in the scope of those recursive realizability interpretations which (like Kleene's 1945-realizability [Kleene, 1945]) can be used to get consistency proofs for "anticlassical" systems (e.g., Intuitionistic Arithmetic enriched by Church's Thesis [Troelstra, 1973]).

As for $(\mathrm{KP}_\vee)$, it is the *Kreisel-Putnam Principle*, a principle well known in the literature of propositional intermediate logics [Kreisel and Putnam, 1957], while $(\mathrm{KP}_\exists)$ (also known in the area of constructivism as (IP) [Troelstra, 1973]) naturally completes the meaning of $(\mathrm{KP}_\vee)$ at the predicate level. A maximal constructive intermediate predicate logic including $(\mathrm{KP}_\vee)$ and $(\mathrm{KP}_\exists)$ is studied in [Avellone et al., 1996].

Finally (wGrz) is a weakened form of the well known *Grzegorczyk Principle* $\forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B$ whose addition to intuitionistic logic gives rise to a well-known intermediate constructive logic [Görnemann, 1971].

The above principles can be expressed by the following pseudo-natural deduction rules:

$$\frac{\Gamma_1 \vdash \forall x \neg\neg A(x) \qquad \Gamma_2 \vdash \forall x(A(x) \vee B)}{\Gamma_1, \Gamma_2 \vdash \forall x A(x) \vee B}\,\mathrm{wGrz} \quad \text{with } x \notin \mathrm{FV}(B)$$

$$\frac{\Gamma \vdash \forall x \neg\neg A(x)}{\Gamma \vdash \neg\neg \forall x A(x)}\,\mathrm{Kur} \qquad \frac{\Gamma, \neg A \vdash B \vee C}{\Gamma \vdash (\neg A \rightarrow B) \vee (\neg A \rightarrow C)}\,\mathrm{KP}_\vee \qquad \frac{\Gamma, \neg A \vdash \exists x B(x)}{\Gamma \vdash \exists x(\neg A \rightarrow B(x))}\,\mathrm{KP}_\exists$$

Now, let $\mathscr{N}\mathscr{D}_{\mathbf{HA}^+}$ be the pseudo-natural deduction calculus obtained by adding the above rules to $\mathscr{N}\mathscr{D}_{\mathbf{HA}}$. In the following we will prove that $\mathscr{N}\mathscr{D}_{\mathbf{HA}^+}$ is uniformly semiconstructive in $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$.

We denote with $\mathrm{R}\textsc{ha}^+$ the union of the generalized rules of Table 3 and of the generalized rules $\textsc{Cut}$ and $\textsc{Subst}$. It is easy to check that $\mathrm{R}\textsc{ha}^+$ is an e-rule for $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$. Let us denote with $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$ the abstract calculus $\mathbb{D}(\mathrm{R}\textsc{ha}^+, \mathrm{Seq}([\Pi]))$.

The proof of uniform semiconstructivity will be carried out using the following notion of evaluation.

Definition 4.1 (Neg**-evaluation)** *Let* Π *be a set of proofs on* $\mathscr{L}_\mathscr{A}$*, and let* Neg *and* A *be a set of closed negated wff's and a wff in the language* $\mathscr{L}_\mathscr{A}$ *respectively.* A *is* Neg-evaluated in Π *iff the following conditions hold:*

(i) Either $A \in \mathrm{Neg}$ *or there exists a proof* $\pi : \Gamma \vdash A \in \Pi$ *with* $\Gamma \subseteq \mathrm{Neg}$*;*

(ii) For every closed instance θA *of* A*, one of the following conditions holds:*

- *(a)* θA *is atomic or negated;*
- *(b)* $\theta A \equiv B \wedge C$*, and both* B *and* C *are* Neg*-evaluated in* Π*;*
- *(c)* $\theta A \equiv B \vee C$*, and either* B *is* Neg*-evaluated in* Π *or* C *is* Neg*-evaluated in* Π*;*
- *(d)* $\theta A \equiv B \rightarrow C$*, and, for every set* Neg' *of closed negated wff's of* $\mathscr{L}_\mathscr{A}$ *such that* $\mathrm{Neg}' \supseteq \mathrm{Neg}$*, if* B *is* Neg'*-evaluated in* Π *then* C *is* Neg'*-evaluated in* Π*;*
- *(e)* $\theta A \equiv \exists x B(x)$*, and* $B(t/x)$ *is* Neg*-evaluated in* Π *for some closed term* t *of* $\mathscr{L}_\mathscr{A}$*;*
- *(f)* $\theta A \equiv \forall x B(x)$*, and, for every closed term* t *of* $\mathscr{L}_\mathscr{A}$*,* $B(t/x)$ *is* Neg*-evaluated in* Π*.*

$\textsc{Id}_1$:	$\vdash x = x$	$\in$	$\textsc{Id}_1(\epsilon)$
$\textsc{Id}_2$:	$\Gamma, \Delta \vdash A(t')$	$\in$	$\textsc{Id}_2(\Gamma \vdash A(t); \Delta \vdash t = t')$
$\textsc{Sum}$:	$\vdash x + 0 = x$	$\in$	$\textsc{Sum}(\epsilon)$
	$\vdash x + Sy = S(x + y)$	$\in$	$\textsc{Sum}(\epsilon)$
$\textsc{Prod}$:	$\vdash x * 0 = 0$	$\in$	$\textsc{Prod}(\epsilon)$
	$\vdash x * Sy = x * y + x$	$\in$	$\textsc{Prod}(\epsilon)$
$\textsc{Rkp}\vee$:	$\Gamma, \Delta \vdash \neg A \to B$	$\in$	$\textsc{Rkp}\vee_1(\Gamma \vdash B; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$ with $\neg A \not\in \Gamma$
	$\Gamma, \Delta \vdash \neg A \to B$	$\in$	$\textsc{Rkp}\vee_1(\Gamma, \neg A \vdash B; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$
	$\Gamma, \Delta \vdash \neg A \to C$	$\in$	$\textsc{Rkp}\vee_2(\Gamma \vdash C; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$ with $\neg A \not\in \Gamma$
	$\Gamma, \Delta \vdash \neg A \to C$	$\in$	$\textsc{Rkp}\vee_2(\Gamma, \neg A \vdash C; \Delta \vdash (\neg A \to B) \vee (\neg A \to C))$
	$\neg B \vdash \neg B$	$\in$	$\textsc{Rkp}\vee_3(\Delta \vdash (\neg A \to \neg B) \vee (\neg A \to C))$
	$\neg C \vdash \neg C$	$\in$	$\textsc{Rkp}\vee_4(\Delta \vdash (\neg A \to B) \vee (\neg A \to \neg C))$
$\textsc{Rkp}\exists$:	$\Gamma, \Delta \vdash \neg A \to B(t)$	$\in$	$\textsc{Rkp}\exists_1(\Gamma \vdash B(t); \Delta \vdash \exists x(\neg A \to B(x)))$ with $\neg A \not\in \Gamma$
	$\Gamma, \Delta \vdash \neg A \to B(t)$	$\in$	$\textsc{Rkp}\exists_1(\Gamma, \neg A \vdash B(t); \Delta \vdash \exists x(\neg A \to B(x)))$
	$\neg B(t) \vdash \neg B(t)$	$\in$	$\textsc{Rkp}\exists_2(\Delta \vdash \exists x(\neg A \to B(x)))$
$\textsc{Rcl}$:	$\Gamma \vdash \forall x A(x)$	$\in$	$\textsc{Rcl}(\Gamma \vdash \forall x \neg\neg A(x))$

Table 3: The generalized rule $\textsc{Rha}^+$

A set Γ of wff's is Neg-evaluated in a set of proofs Π if every wff $A \in \Gamma$ is Neg-evaluated in Π. The following results, which can be easily proved, will be needed.

Proposition 4.2 *Let* Neg *be a set of closed negated wff's, let* Π *be a set of proofs and let* A *be a wff.*

1. *If* A *is* Neg*-evaluated in* Π*, then* A *is* Neg$'$*-evaluated in* Π *for every set of closed negated wff's* Neg$'$ *including* Neg.

2. *If* Π *is* $\textsc{Cut}$*-closed,* $\neg H$ *is a closed wff* Neg*-evaluated in* Π*, and* A *is* $\mathrm{Neg} \cup \{\neg H\}$*-evaluated in* Π*, then* A *is* Neg*-evaluated in* Π.

Lemma 4.3 *Let* Π *be any recursive set of proofs of* $\mathscr{N}\mathscr{D}_{\mathbf{HA}^+}$ *and let* Neg *be a set of closed negated wff's of* $\mathscr{L}_{\mathscr{A}}$*. For any proof* $\pi : \Gamma \vdash H$ *belonging to the closure under substitution of* $[\Pi]$*, if* Γ *is* Neg*-evaluated in* $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$*, then* H *is* Neg*-evaluated in* $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$.

Proof: Since π belongs to the closure under substitution of $[\Pi]$, there exist a proof $\pi' : \Gamma' \vdash H' \in [\Pi]$ and a substitution θ such that $\theta\Gamma' \vdash \theta H' \equiv \Gamma \vdash H$. Thus, by definition, $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$ contains a proof of the sequent $\Gamma' \vdash H'$ and, since it is $\textsc{Subst}$-closed, it also contains a proof $\tau' : \Gamma \vdash H$. Let $\Gamma = \Delta_0 \cup \{H_1, \ldots, H_n\}$, where $\Delta_0 = \Gamma \cap \mathrm{Neg}$. Since Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$ contains proofs $\tau_1 : \Delta_1 \vdash H_1, \ldots, \tau_n : \Delta_n \vdash H_n$ with $\Delta_1 \cup \cdots \cup \Delta_n \subseteq \mathrm{Neg}$. Let $\Delta^* = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_n$; by repeatedly applying the $\textsc{Cut}$ rule to the proofs $\tau', \tau_1, \ldots, \tau_n$, we can construct in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$ a proof $\tau^* : \Delta^* \vdash H$.

The proof of Point (ii) goes on by induction on depth(π). If depth(π) = 0, the only rule which occurs in π is either an assumption introduction or one of the zero-premises rules corresponding to the axioms for identity, sum and product. In the former case the assertion trivially follows. In the latter case, the wff H is atomic; hence, by Point (i) and the closure under SUBST of $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, we immediately get the assertion. To prove the induction step, we proceed by cases according to the last rule applied in π. Here we only treat the representative cases of the rules Ind, wGrz and $\mathrm{KP}_\vee$.

Induction Rule.

$$\pi : \Gamma \vdash H \equiv \frac{\pi_1 : \Gamma_1 \vdash B(0) \quad \pi_2 : \Gamma_2, B(p) \vdash B(S(p))}{\Gamma_1, \Gamma_2 \vdash B(x)}\,\mathrm{Ind}$$

First of all, the reader can easily prove that, given a closed term t of $\mathscr{L}_{\mathscr{A}}$, $B(t)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$ iff $B(S^n 0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, where S^n0 is the canonical form of t in **HA** (this derives from the closure of $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$ with respect to the rules ID_1, ID_2, SUM and PROD). Therefore, to prove the assertion it suffices to show that $\theta B(S^h 0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, for every closed substitution θ and every $h \geq 0$. We proceed by a secondary induction on h. Since $\Gamma_1 \subseteq \Gamma$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, by the principal induction hypothesis applied to π_1 we get that $\theta B(0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Let us suppose that $\theta B(S^h 0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, with $h \geq 0$; since the proof $\theta[S^h0/p]\pi_2 : \theta\Gamma_2, \theta B(S^h0/p) \vdash \theta B(S^{h+1}0/p)$ belongs to the closure under substitution of $[\Pi]$, we can apply the principal induction hypothesis and state that $B(S^{h+1}0)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$.

Rule wGrz.

$$\pi : \Gamma \vdash H \equiv \frac{\pi_1 : \Gamma_1 \vdash \forall x \neg\neg B(x) \quad \pi_2 : \Gamma_2 \vdash \forall x (B(x) \vee C)}{\Gamma_1, \Gamma_2 \vdash \forall x B(x) \vee C}\,\mathrm{wGrz}$$

Let θ be a closed substitution; we must prove that one between the wff's $\theta\forall x B(x)$ and θC is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Let us suppose that θC is not Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Since, by induction hypothesis on π_2, $\theta\forall x(B(x) \vee C)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, we deduce that, for every closed term t of $\mathscr{L}_{\mathscr{A}}$, $\theta B(t/x)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. To prove that $\theta\forall x B(x)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, we only need to show that $\theta\forall x B(x)$ is provable from Neg in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. By induction hypothesis on the proof π_1, a sequent $\Gamma' \vdash \theta\forall x \neg\neg B(x)$, with $\Gamma' \subseteq \mathrm{Neg}$, is provable in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$; since the latter set of proofs is RCL-closed, we get that also $\Gamma' \vdash \theta\forall x B(x)$ belongs to $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$.

Rule $\mathrm{KP}_\vee$.

$$\pi : \Gamma \vdash H \equiv \frac{\pi_1 : \Gamma, \neg B \vdash C \vee D}{\Gamma \vdash (\neg B \to C) \vee (\neg B \to D)}\,\mathrm{KP}_\vee$$

We must prove that, for every closed substitution θ, one between the wff's $\theta(\neg B \to C)$ and $\theta(\neg B \to D)$ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Since Γ is Neg-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, we have that $\theta\Gamma \cup \{\theta\neg B\}$ is $\mathrm{Neg} \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Then, by the induction hypothesis on the proof $\theta\pi_1$, either θC or θD is $\mathrm{Neg} \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Let us assume that θC is the evaluated wff; this implies that there exists a proof $\tau : \Delta \vdash \theta C \in \mathbb{D}_{\mathbf{HA}^+}([\Pi])$, with $\Delta \subseteq \mathrm{Neg} \cup \{\theta\neg B\}$. We remark that, if $\theta C \in \mathrm{Neg} \cup \{\theta\neg B\}$, we can construct τ as follows

$$\cfrac{\cfrac{\tau^* : \Delta^* \vdash (\neg B \to C) \vee (\neg B \to D)}{\theta\Delta^* \vdash \theta(\neg B \to C) \vee \theta(\neg B \to D)}\,\textsc{Subst}}{\theta C \vdash \theta C}\,\textsc{Rkp}\vee_3$$

where $\tau^* : \Delta^* \vdash H$ is defined in the proof of Point (i). Hence, the proof

$$\dfrac{\tau : \Delta \vdash \theta C \qquad \dfrac{\tau^* : \Delta^* \vdash (\neg B \to C) \vee (\neg B \to D)}{\theta\Delta^* \vdash \theta(\neg B \to C) \vee \theta(\neg B \to D)}\ \textsc{Subst}}{\Delta \setminus \{\theta\neg B\},\ \theta\Delta^* \vdash \theta(\neg B \to C)}\ \textsc{Rkp}\vee_1$$

belongs to $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, and this proves Point (i) of Definition 4.1 for $\theta(\neg B \to C)$. To prove Point (ii) for this wff, let us suppose that $\theta\neg B$ is Neg'-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, with $\mathrm{Neg} \subseteq \mathrm{Neg}'$. We already know that θC is $\mathrm{Neg} \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, and hence, by Point (1) of Proposition 4.2, it is also $\mathrm{Neg}' \cup \{\theta\neg B\}$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Since $\theta\neg B$ is Neg'-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, by Point (2) of Proposition 4.2 it follows that θC is Neg'-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. This concludes the proof. □

Corollary 4.4 *Let* Π *be any recursive set of proofs of* $\mathscr{ND}_{\mathbf{HA}^+}$. *Then the set* $\mathrm{Theo}([\Pi])$ *is semiconstructive in* $\mathrm{Theo}(\mathbb{D}_{\mathbf{HA}^+}([\Pi]))$.

Proof: If $A \vee B$ is a closed wff in $\mathrm{Theo}([\Pi])$, then there exists a proof $\pi : \vdash A \vee B$ in the closure under substitution of $[\Pi]$. Since the empty set of premises is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$, by Lemma 4.3 it follows that $A \vee B$ is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. Thus, at least one of the wff's A and B is $\emptyset$-evaluated in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$; by Definition 4.1, this means that one between the sequents $\vdash A$ and $\vdash B$ is provable in $\mathbb{D}_{\mathbf{HA}^+}([\Pi])$. The proof of the (wEd) property is similar. □

Since $\textsc{Rha}^+$ is an e-rule for $\mathscr{ND}_{\mathbf{PA}}$ and $\mathscr{ND}_{\mathbf{HA}^+}$ is uniformly embedded in $\mathscr{ND}_{\mathbf{PA}}$, by the previous corollary we get:

Theorem 4.5 $\mathscr{ND}_{\mathbf{HA}^+}$ *is a uniformly semiconstructive calculus in* $\mathscr{ND}_{\mathbf{PA}}$.

As a consequence of the previous theorem $\mathbf{HA}^+ = \mathrm{Theo}(\mathscr{ND}_{\mathbf{HA}^+})$ is an uniformly semiconstructive **HA**-system. To conclude the presentation of this example, we will prove that $\mathbf{HA}^+$ cannot be extended into a recursively enumerable and constructive **T**-system, with **T** any theory including **HA**.

Theorem 4.6 *There exists no consistent and recursively axiomatizable constructive* **T**-*system* **S** *such that* $\mathbf{HA} \subseteq \mathbf{T}$ *and* $\mathbf{HA}^+ \subseteq \mathbf{S}$.

Proof: Let **S** be a constructive recursively axiomatizable **T**-system including $\mathbf{HA}^+$ (with $\mathbf{HA} \subseteq \mathbf{T}$). Let $p(x)$ be a unary recursively enumerable but not recursive predicate. By the Normal Form Theorem, there exists a primitive recursive binary predicate $q(x, y)$ such that $p(x) \leftrightarrow \exists y q(x, y)$. Now, by well known representability results, there exists a wff $A(x, y)$ with two free variables of $\mathscr{L}_{\mathscr{A}}$ which strongly represents the predicate $q(x, y)$ in **HA**; this means that $\forall x \forall y (A(x, y) \vee \neg A(x, y)) \in \mathbf{HA}$, and, for every $a, b \in \mathbf{N}$, denoting with $\widetilde{a}$ and $\widetilde{b}$ the corresponding numerals, if $q(a, b)$ is true then $A(\widetilde{a}, \widetilde{b}) \in \mathbf{HA}$, if $q(a, b)$ is false then $\neg A(\widetilde{a}, \widetilde{b}) \in \mathbf{HA}$. Moreover, let G be a closed wff of $\mathscr{L}_{\mathscr{A}}$ such that $G \notin \mathbf{S}$ and $\neg G \notin \mathbf{S}$ (such a wff exists by the intuitionistic version of Gödel Incompleteness Theorem [Troelstra, 1973]). Let us consider the wff $H(x) \equiv \exists y A(x, y) \vee \forall y (\neg A(x, y) \vee (G \vee \neg G))$. It is easy to check that $H(x)$ is provable in $\mathbf{HA}^+$. We will show that one can effectively decide whether $p(k)$ holds or not, for every natural number k. Indeed, since **S** is constructive and recursively axiomatizable, there is a terminating effective procedure which, for every input $k \in \mathbf{N}$, outputs either a **S**-proof of $\exists y A(\widetilde{k}, y)$ or a **S**-proof of $\forall y (\neg A(\widetilde{k}, y) \vee (G \vee \neg G))$.

If $\exists y A(\widetilde{k}, y) \in \mathbf{S}$, then, by the constructivity of $\mathbf{S}$, $A(\widetilde{k}, \widetilde{b}) \in \mathbf{S}$ for some numeral $\widetilde{b}$; this implies that $\exists y q(k, y)$ holds. On the other hand, if $\forall y(\neg A(\widetilde{k}, y) \vee (G \vee \neg G)) \in \mathbf{S}$, since, by the hypotheses on the wff G, $G \vee \neg G \notin \mathbf{S}$, we deduce that $\neg A(\widetilde{k}, \widetilde{b}) \in \mathbf{S}$, for every $b \in \mathbf{N}$; this implies that $\exists y q(k, y)$ does not hold. Since $p(x) \leftrightarrow \exists y q(x, y)$, we have that $p(x)$ is recursive, against the assumptions. □

4.2 The uniformly semiconstructive HA-system HA^{++}

Let us consider the **HA**-system $\mathbf{HA}^{++}$ obtained by adding to Intuitionistic Arithmetic **HA** the following principles

$$\begin{array}{ll} \text{(DT)} & \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B) \\ \text{(Mk)} & \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \to \exists x A(x) \end{array}$$

where (Mk) is the well known *Markov Principle* (see, e.g., [Troelstra, 1973] while (DT) is a predicate extension of the propositional principle characterizing the intermediate propositional logic whose frames have depth at most 2 [Chagrov and Zakharyaschev, 1997].

$\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}}$ will denote the calculus for $\mathbf{HA}^{++}$ obtained by adding to the calculus $\mathscr{N}\mathscr{D}_{\mathbf{HA}}$ the rules

$$\frac{}{\vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)}\,\text{DT} \qquad \frac{\Gamma, \neg\neg\exists x A(z) \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma, \neg\neg\exists x A(x) \vdash \exists x A(x)}\,\text{Mk}$$

Now, we denote with RHA^{++} the union of the generalized rules Cut and Subst, of the generalized rules Id, Sum, Prod of Table 3, and of the generalized rules of Table 4. It is easy to check that RHA^{++} is an e-rule for $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$.

RDT$_1$:	$\vdash \exists x A(x)$	$\in$	RDT$_1$($\vdash A(t); \vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)$)
RDT$_2$:	$\vdash \forall x(A(x) \to B \vee \neg B)$	$\in$	RDT$_2$($\vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)$)
RDT$_3$:	$\vdash A(x) \to B \vee \neg B$	$\in$	RDT$_3$($\vdash \exists x A(x) \vee \forall x(A(x) \to B \vee \neg B)$)

Table 4: The generalized rule RHA^{++}

Let us denote with $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$ the abstract calculus $\mathbb{D}(\text{RHA}^{++}, \text{Seq}([\Pi]))$. The proof of uniform semiconstructivity of $\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}}$ in $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$ can be carried out along the lines of the proof given in the previous section, but using the following simpler notion of evaluation.

Definition 4.7 (Closed evaluation) *Let* Π *be a set of proofs on* $\mathscr{L}_{\mathscr{A}}$ *and let* A *be a wff in the language* $\mathscr{L}_{\mathscr{A}}$. A is evaluated in Π *iff the following conditions hold:*

(i) There is a proof $\pi : \vdash A \in \Pi$*;*

(ii) For every closed instance θA *of* A*, one of the following conditions holds:*

(a) θA *is atomic or negated;*

(b) $\theta A \equiv B \wedge C$*, and both* B *and* C *are evaluated in* Π*;*

(c) $\theta A \equiv B \vee C$*, and either* B *is evaluated in* Π *or* C *is evaluated in* Π*;*

(d) $\theta A \equiv B \rightarrow C$*, and either* B *is not evaluated in* Π *or* C *is evaluated in* Π*;*

(e) $\theta A \equiv \exists x B(x)$*, and* $B(t/x)$ *is evaluated in* Π *for some closed term* t *of* $\mathscr{L}_{\mathscr{A}}$*;*

(f) $\theta A \equiv \forall x B(x)$*, and, for every closed term* t *of* $\mathscr{L}_{\mathscr{A}}$*,* $B(t/x)$ *is evaluated in* Π*.*

A set Γ of wff's is evaluated in a set of proofs Π if every wff $A \in \Gamma$ is evaluated in Π. Hence the main lemma is:

Lemma 4.8 *Let* Π *be any recursive set of proofs of* $\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}}$*. For any proof* $\pi : \Gamma \vdash H$ *belonging to the closure under substitution of* $[\Pi]$*, if* Γ *is evaluated in* $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$*, then* H *is evaluated in* $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$*.*

Proof: The proof is similar to the one of Lemma 4.3; we only have to analyze the cases corresponding to the rules DT and Mk. In the former case, $\mathrm{depth}(\pi) = 0$, Γ is empty and $H \equiv \exists x A(x) \vee \forall x(A(x) \rightarrow B \vee \neg B)$. Let us consider any closed instance θH of this wff and let us suppose that $\theta \exists x A(x)$ is not evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$; we prove that $\theta \forall x(A(x) \rightarrow B \vee \neg B)$ is evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$. Since $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$ is SUBST-closed and RDT_2-closed, it contains a proof of $\theta \forall x(A(x) \rightarrow B \vee \neg B)$. Let t be any closed term of $\mathscr{L}_{\mathscr{A}}$; by the closure of $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$ under the rules RDT_3 and SUBST, a proof of $\theta(A(t) \rightarrow B \vee \neg B)$ belongs to $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$. Moreover, $\theta A(t)$ is not evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$ (otherwise, by the RDT_1-closure of $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$, $\theta \exists x A(x)$ would be evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$), thus $\theta(A(t) \rightarrow B \vee \neg B)$ is evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$, and this concludes the proof.

Markov Rule.

$$\frac{\pi_1 : \Gamma', \neg\neg\exists x A(x) \vdash \forall x(A(x) \vee \neg A(x))}{\Gamma', \neg\neg\exists x A(x) \vdash \exists x A(x)}\,\mathrm{Mk}$$

Let us consider a closed instance $\theta \exists x A(x)$ of $\exists x A(x)$. By induction hypothesis on the proof $\theta \pi_1$, we have that $\theta \forall x(A(x) \vee \neg A(x))$ is evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$. Hence, for every closed term t of $\mathscr{L}_{\mathscr{A}}$, $\theta A(t/x) \vee \neg \theta A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$. Let us suppose that, for every closed term t of $\mathscr{L}_{\mathscr{A}}$, $\neg \theta A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$; then, for every closed term t of $\mathscr{L}_{\mathscr{A}}$, $\vdash \theta \neg A(t/x)$ is provable in $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$. On the other hand, $\theta \neg\neg\exists x A(x)$ evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$ implies that $\vdash \theta \exists x A(x)$ is provable in $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$. But, since the standard model of arithmetic is reachable (that is every element of its domain is denoted by a closed term of $\mathscr{L}_{\mathscr{A}}$), we get a contradiction. Hence, there must exist a closed term t of $\mathscr{L}_{\mathscr{A}}$ such that $\theta A(t/x)$ is evaluated in $\mathbb{D}_{\mathbf{HA}^{++}}([\Pi])$. □

From the previous lemma, we get:

Corollary 4.9 *Let* Π *be any recursive set of proofs of* $\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}}$*. Then the set* $\mathrm{Theo}([\Pi])$ *is semiconstructive in* $\mathrm{Theo}(\mathbb{D}_{\mathbf{HA}^{++}}([\Pi]))$*.*

Since RHA^{++} is an e-rule for $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$ and $\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}}$ is uniformly embedded in $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$, by the previous corollary we get:

Theorem 4.10 $\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}}$ *is an uniformly semiconstructive calculus in* $\mathscr{N}\mathscr{D}_{\mathbf{PA}}$*.*

As a consequence of this theorem $\mathbf{HA}^{++} = \mathrm{Theo}(\mathscr{N}\mathscr{D}_{\mathbf{HA}^{++}})$ is an uniformly semiconstructive **HA**-system.

We point out that the well known *Scott Principle* (St) $((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A$ [Chagrov and Zakharyaschev, 1997] is derivable from (DT). On the other hand, the addition of both (St) and ($\mathrm{KP}_{\exists}$) to **HA** gives rise to a **HA**-system which is not semiconstructive [Ferrari et al., 1999b]; this implies that there is no semiconstructive **HA**-system which contains both the semiconstructive **HA**-systems $\mathbf{HA}^{+}$ and $\mathbf{HA}^{++}$ (in particular, $\mathbf{HA}^{+} \not\subseteq \mathbf{HA}^{++}$ and $\mathbf{HA}^{++} \not\subseteq \mathbf{HA}^{+}$). We remark that we can add to $\mathbf{HA}^{++}$ the rule Kur of § 4.1 without affecting its uniform semiconstructivity (and without extending the generalized rule $\mathrm{R}_{\mathrm{HA}^{++}}$). However, we can prove that $\mathbf{HA}^{++}$ cannot be extended into a recursively enumerable and constructive **T**-system with $\mathbf{HA} \subseteq \mathbf{T}$.

Theorem 4.11 *There exists no consistent and recursively axiomatizable constructive* **T**-*system* **S** *such that* $\mathbf{HA} \subseteq \mathbf{T}$ *and* $\mathbf{HA}^{++} \subseteq \mathbf{S}$.

Proof: Let **S** be a recursively axiomatizable and constructive **T**-system including $\mathbf{HA}^{++}$ (with $\mathbf{HA} \subseteq \mathbf{T}$). We will show that, for every closed wff A, one can decide whether $A \in \mathbf{S}$ or not. Indeed, let G be a closed wff of $\mathscr{L}_{\mathscr{A}}$ such that $G \notin \mathbf{S}$ and $\neg G \notin \mathbf{S}$. Since **S** is constructive and recursively axiomatizable and, for every closed wff A, $A \vee (A \to G \vee \neg G) \in \mathbf{HA}^{++}$, there is a terminating effective procedure which, taking any closed wff A of $\mathscr{L}_{\mathscr{A}}$ as an input, outputs either a **S**-proof of A or a **S**-proof of $A \to G \vee \neg G$. If $A \to G \vee \neg G \in \mathbf{S}$, by the choice of G and the constructivity of **S**, $A \notin \mathbf{S}$. Hence, the set of theorems of **S** is recursive, against the Intuitionistic version of Church's Theorem. □

5 Conclusions

We conclude by remarking that the extraction mechanism described in this paper is rather general. It can be applied to a wide family of **T**-systems including theories with isoinitial model (formalizing Abstract Data Types according to [Miglioli et al., 1994]) and logical and mathematical principles of interest in the framework of program synthesis and verification (as those studied in [Thompson, 1991; Avellone et al., 1999]). Finally, we want to remark that the notion of uniformly semiconstructive formal system does not collapse in the notion of semiconstructive formal system; in fact in [Ferrari et al., 1999b] the authors exhibit a formal system obtained by adding to Intuitionistic Arithmetic a single axiom schema which is semiconstructive but is not uniformly semiconstructive.

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